Algebraic Geometry II - Notes by Bastiaan Cnossen

Lecture 1 - 09-04-18

The main topic of this lecture course will be ~~the~~ cohomology theory of (quasi-) coherent sheaves of modules.

Let $\mathcal{C}_X$ be the sheaf of $\mathbb{C}$-valued continuous functions on X and $\mathcal{C}_X^*$ be the sheaf of $\mathbb{C}^*$-valued — " — — " — — " —.

We have a short exact sequence

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{\cdot 2\pi i} \mathcal{C}_X \xrightarrow{\exp} \mathcal{C}_X^* \longrightarrow 0$$

$$f(\cdot) \longmapsto \exp(f(\cdot))$$

($\underline{\mathbb{Z}}$ is the constant sheaf)

Unfortunately, the sequence is no longer exact at global sections, since there is no complex logarithm. We get a long exact sequence

$$0 \longrightarrow \underbrace{\underline{\mathbb{Z}}}_{= H^0(X, \underline{\mathbb{Z}})} \longrightarrow \underbrace{\mathcal{C}_X(X)}_{= H^0(X, \mathcal{C}_X)} \xrightarrow{\exp} \underbrace{\mathcal{C}_X^*(X)}_{H^0(X, \mathcal{C}_X^*)}$$

$$\xrightarrow{d} H^1(X, \underline{\mathbb{Z}}) \rightarrow H^1(X, \mathcal{C}_X) \longrightarrow H^1(X, \mathcal{C}_X^*)$$

$$\xrightarrow{d} H^2(X, \underline{\mathbb{Z}}) \longrightarrow H^2(X, \mathcal{C}_X) \longrightarrow H^2(X, \mathcal{C}_X^*) \longrightarrow \cdots$$

Derived functors (Grothendieck, Tohoku)

We won't go through the general definition of derived functors, but will discuss some properties.

The homology groups above are actually derived functors of the functor of global sections.

Section 1: Cohomology of q.c. sheaves of modules

1.1 Recollection of basic definitions and results

Def 1.1.1 (Pre-) scheme. (Subcategory of category of locally ringed spaces)

- Prescheme: A locally ringed space $(X, \mathcal{O}_X)$ which locally has the form $\operatorname{Spec} R$.
- Scheme: A prescheme X s.t. for any prescheme T and any pair of morphisms $a, b: T \to X$, the equalizer $\ker(T \overset{a}{\underset{b}{\rightrightarrows}} X)$ is closed sub-prescheme of T, or equivalently a prescheme X s.t. $X \xrightarrow{\Delta} X \times X$ is a closed immersion

(So a scheme is a separated prescheme)

Remember that $\ker(T \overset{a}{\underset{b}{\rightrightarrows}} X)$ has as underlying topological space

$$K = \{t \in T \mid a(t) = b(t) =: x \text{ and } a^* = b^* : k(x) \longrightarrow k(t)\}$$

Here $k(t) = \mathcal{O}_{T,t}/\mathfrak{m}_t$ is the residue field of the local ring $\mathcal{O}_{T,t}$.

Also remember:

$\xi : X \longrightarrow Y$ is separated

$\Leftrightarrow \ker(T \overset{a}{\underset{b}{\rightrightarrows}} X) \longrightarrow T$ is a closed subprescheme when $a, b : T \longrightarrow X$ are s.t. $\xi a = \xi b$

$\Leftrightarrow X \xrightarrow{\Delta} X \times_Y X$ is closed.

<u>Proposition 1.1.1</u>: If U and V are affine open subsets of a scheme X, then $U \cap V$ is affine.

<u>Definition 1.1.2</u> a) A <u>pointed</u> category is a category with an initial and a final object such that the canonical morphism from the initial object to the final object is an isomorphism.

b) An <u>additive</u> category is a pointed category which has products (i.e. fibre products over the final object) and coproducts (i.e. dual fibre products w.r.t. the initial object) and s.t. the canonical morphism $A \sqcup B \longrightarrow A \times B$ is always an isomorphism, and s.t. the resulting addition law on $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is always a group structure.

<u>Remark</u>: a) Let $\mathcal{A}$ be a pointed category and X, Y two objects. The zero morphism $0 \in \mathrm{Hom}(X, Y)$ is the composition $X \longrightarrow * \longrightarrow Y$

b) The product $X \times Y$ comes with morphisms $X \xleftarrow{p_X} X \times Y \xrightarrow{p_Y} Y$ s.t. $X \times Y$ satisfies a u.p. on the right,

whereas the coproduct $X \sqcup Y$ comes with morphisms $X \xrightarrow{i_X} X \sqcup Y \xleftarrow{i_Y} Y$ and has a left u.p.

There is thus a unique morphism $X \sqcup Y \xrightarrow{c} X \times Y$ s.t.

$$p_X c\, i_X = \mathrm{id}_X \qquad p_X c\, i_Y = 0$$

$$p_Y c\, i_Y = \mathrm{id}_Y \qquad p_Y c\, i_X = 0$$

In other words, $X \sqcup Y \xrightarrow{p_X c} X$ is the unique morphism whose composition with i_X is id_X and with i_Y is 0, and similar for $X \sqcup Y \xrightarrow{p_Y c} Y$.

c) For abelian groups, or modules over a ring, $X \amalg Y$ and $X \prod Y$ are both given by $X \times Y = \{(x,y) \mid x \in X, y \in Y\}$, with the component-wise operations, where

$$p_X(x,y) = x \qquad i_X(x) = (x,0)$$
$$p_Y(x,y) = y. \qquad i_Y(y) = (0,y)$$

d) For any additive category, it follows that finite products and coproducts exist and that

$$\prod_{i=1}^{n} X_i \cong \coprod_{i=1}^{n} X_i \qquad \text{(canonical isomorphism)}$$

e) If X and T are arbitrary objects, and $a,b : X \to T$ are morphisms, then the <u>sum</u> $a+b : X \to T$ is defined as the composition

$$X \xrightarrow{\Delta_X} X \prod X \cong X \amalg X \xrightarrow{(a,b)} T \amalg T \xrightarrow{\nabla_T} T,$$

where the dual diagonal $T \amalg T \xrightarrow{\nabla_T} T$ is the unique morphism whose compositions with the two inclusions $i_{1,2} : T \to T \amalg T$ equal id_T.

The zero object $0 : X \to T$ is a neutral element for this addition, but the full group axioms (existence of $-a$) have to be imposed as an additional condition.

For the category of abelian groups this is just usual addition.

f) Typically, we denote $\prod_{i=1}^{n} X_i$ and $\coprod_{i=1}^{n} X_i$ by $\bigoplus_{i=1}^{n} X_i$ if the category is additive.

<u>Remark</u> Examples of additive categories:

a) Modules over a given ring R (in particular, abelian groups)

b) Sheaves of modules

c) Banach spaces with bounded linear maps as morphisms
$(A \oplus B = \{(a,b) \in A \times B\}$ with $\max\{\|a\|, \|b\|\}$ or $\|a\| + \|b\|$ as norm

d) Free or projective modules over a ring R

<u>Remark 1.1.1</u>: For kernels and cokernels in an additive category, the following universal properties are imposed:

- $$\mathrm{Hom}(T, \mathrm{Ker}(A \xrightarrow{\alpha} B)) \xrightarrow{\cong} \{f \in \mathrm{Hom}(T,A) \mid \alpha f = 0\}$$
$$(T \xrightarrow{t} K) \longmapsto f = it$$

- $$\mathrm{Hom}(\mathrm{Coker}(A \xrightarrow{\beta} B), T) \xrightarrow{\cong} \{g \in \mathrm{Hom}(B,T) \mid g\beta = 0\}$$
$$(\mathrm{Coker}(\beta) \xrightarrow{t} T) \longmapsto g = t\pi$$

Thus $\mathrm{Ker}(A \xrightarrow{\alpha} B) = \mathrm{Ker}(A \overset{\alpha}{\underset{0}{\rightrightarrows}} B)$ and

$\mathrm{Coker}(A \xrightarrow{\alpha} B) = \mathrm{Coker}(A \overset{\alpha}{\underset{0}{\rightrightarrows}} B)$.

<u>Definition</u>: An <u>abelian</u> category is an additive category such that the following equivalent conditions hold:

$\boxed{\text{Note: incomplete definition!}}$

- it has kernels and cokernels
- it has equalizers and coequalizers
- it has fibre products and dual fibre product
- it has finite limits and colimits

($\mathrm{Ker}(A \overset{\alpha}{\underset{\beta}{\rightrightarrows}} B) = \mathrm{Ker}(A \xrightarrow{\alpha-\beta} B)$, and

$$\varprojlim_C F = \mathrm{Ker}\Big(\prod_{x\in \mathrm{Ob}C} F(x) \rightrightarrows \prod_{\xi : X\to Y\in \mathrm{Mor}(C)} F(Y)\Big)$$

<u>Definition</u> A morphism $i: A \to B$ is an <u>effective monomorphism</u> if the following equivalent conditions hold:

a) The map

$$\mathrm{Hom}(T,A) \xrightarrow{\sim} \{f\in \mathrm{Hom}(T,B) \mid \alpha f = \beta f \text{ if } B \overset{\alpha}{\underset{\beta}{\rightrightarrows}} S \text{ is any pair s.t. } \alpha i = \beta i\}$$

$$t \longmapsto f = it$$

is a bijection

b) (If category has finite colimits)

f is the equalizer of an appropriate pair of morphisms.

c) (Additive categories with cokernels) ⇓

f is the kernel of an appropriat morphism

$\tilde{c}$) (—— " ——)

f is the kernel of its cokernel

Dually, $p: A \to B$ is an <u>effective epimorphism</u> if:

a) $$\mathrm{Hom}(B,T) \xrightarrow{\sim} \{\varphi \in \mathrm{Hom}(A,T) \mid \varphi\alpha = \varphi\beta \text{ whenever } S \overset{\alpha}{\underset{\beta}{\rightrightarrows}} A \text{ is s.t. } p\alpha = p\beta\}$$

$$\gamma \longmapsto \varphi = \gamma p$$

b) (If finite limits exist)

f is coequalizer of some pair of morphisms

c) ...

$\tilde{c}$) (Additive categories with kernels)

p is the cokernel of its kernel

d) In the dual category, $p^{op}: B^{op} \to A^{op}$ is an effective epimorphism.

Definition: An abelian category is an additive category with kernels and cokernels and such that
a) Any monomorphism is effectively monomorphism
b) Any epimorphism is effectively epi
c) Any morphism that is both mono and epi is an iso

(c) follows from a) and b).)

The categories of modules or sheaves of modules are an abelian category.
However, Banach spaces or projective modules over most rings are not abelian categories.

Example 1.1.1 (Sheaves of modules form an abelian category)
a) 0 is a common initial and final object.
b) A direct sum of A and B is

$$(A \oplus B)(U) = \{(a,b) \mid a \in A(U),\ b \in B(U)\}$$

with the component-wise module operations, and the obvious projections $A \xleftarrow{p} A \oplus B \xrightarrow{q} B$ and embeddings $A \xrightarrow{i} A \oplus B \xleftarrow{j} B$.

c) If $A \xrightarrow{\alpha} T \xleftarrow{\beta} B$ are given, then

$$A \oplus B \xrightarrow{(\alpha,\beta)} T \quad : \quad (a,b) \in (A \oplus B)(U) \longmapsto \alpha(a) + \beta(b)$$

verifies the universal property of the coproduct for $A \oplus B$.
Similarly, $A \oplus B$ is a product, with for $A \xleftarrow{\alpha} T \xrightarrow{\beta} B$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} : T \longrightarrow A \oplus B \quad : \quad t \mapsto (\alpha(t), \beta(t))$$

Also, $id_{A \oplus B}$ satisfies the constraints.
Thus, it is an additive category.

Lecture 3 - 16-04-18

Recall the sheaf axiom for a presheaf G:
for all open covers $U = \bigcup_{\lambda \in \Lambda} U_\lambda$, there is an isomorphism

$$G(U) \xrightarrow{\cong} \{(g_\lambda)_{\lambda \in \Lambda} \in \prod_{\lambda \in \Lambda} G(U_\lambda) \mid g_\lambda|_{U_{\lambda\sigma}} = g_\sigma|_{U_{\lambda\sigma}}\}$$

in other words, there is an exact sequence

$$0 \longrightarrow G(U) \xrightarrow{\cdot|_{U_\lambda}} \prod_{\lambda \in \Lambda} G(U_\lambda) \xrightarrow{g_\lambda|_{U_{\lambda\sigma}} - g_\sigma|_{U_{\lambda\sigma}}} \prod_{\lambda\sigma \in \Lambda} G(U_{\lambda\sigma})$$

The <u>stalk</u> of $\mathcal{G}$ at a point $x \in X$ is

$$\mathcal{G}_x = \varinjlim_{\substack{x\in U\\ U \text{ open}}} \mathcal{G}(U) = \{(g,U) \mid U \subseteq X \text{ open}, x \in U, g \in \mathcal{G}(U)\}/\sim$$

where $(g,U) \sim (h,V)$ iff for some $x \in W \subseteq U \cap V$, $g|_W = h|_W$

The <u>sheafification</u> of a presheaf $\mathcal{G}$ is given by

$$\mathrm{Sheaf}(\mathcal{G})(U) = \left\{ (g_x) \in \prod_{x\in U} \mathcal{G}_x \;\middle|\; \begin{array}{l}\text{for all } x\in U \text{ there is an open nbhd}\\ V \text{ and } \gamma \in \mathcal{G}(V) \text{ s.t. for all } y \in U\cap V,\\ g_y \text{ is the image of } \gamma \text{ under } \mathcal{G}(V) \to \mathcal{G}_y\end{array}\right\}$$

When $\mathcal{G}$ is already a sheaf, the inclusion

$$\mathcal{G} \longrightarrow \mathrm{Sheaf}(\mathcal{G})$$
$$g \longmapsto (g_x)_{x\in U}$$

is an isomorphism of ~~presheaves~~ sheaves, i.e. $\mathcal{G} \cong \mathrm{Sheaf}(\mathcal{G})$.

<u>Kernels in the category of sheaves of modules</u>

For $f: \mathcal{M} \longrightarrow \mathcal{N}$, define $\mathrm{Ker}(f)$ by

$$\mathrm{Ker}(f)(U) = \ker(f(U): \mathcal{M}(U) \longrightarrow \mathcal{N}(U))$$

Then $\mathcal{K} \longrightarrow \mathcal{M}$ is mono, as $\mathcal{K}(U) \to \mathcal{M}(U)$ is injective for every open set U.

If $\tau: \mathcal{T} \longrightarrow \mathcal{M}$ is a morphism s.t. $f\tau = 0$, then for every $t \in \mathcal{T}(U)$, we have $f(\tau(t)) = 0$, so $\tilde{\tau}(t) := \tau(t) \in \mathcal{K}(U)$, and $\tau = k \cdot \tilde{\tau}$, where $k: \mathcal{K} \longrightarrow \mathcal{M}$ is the inclusion.

So $k: \mathcal{K} \longrightarrow \mathcal{M}$ is indeed a kernel of f.

It is a consequence of the exactness of $\varinjlim$ that

$$\mathcal{K}_x \xrightarrow{\cong} \ker(\mathcal{M}_x \xrightarrow{f_x} \mathcal{N}_x)$$

In an additive category, a morphism $i: \mathcal{M} \longrightarrow \mathcal{N}$ is a <u>monomorphism</u> iff $\ker(i) = 0$.

So in our example above, indeed

$$\begin{aligned} f: \mathcal{M} \to \mathcal{N} \text{ is mono} &\iff \ker(f) = 0 \\ &\iff \ker(f)(U) = 0 \quad \text{for all open } U \\ &\iff f(U): \mathcal{M}(U) \to \mathcal{N}(U) \text{ is injective.} \\ &\iff f_x: \mathcal{M}_x \longrightarrow \mathcal{N}_x \text{ is injective } \forall x \in X. \end{aligned}$$

For f being an epimorphism, being surjective on all open U is sufficient but is too strong. As a morphism of sheaves is uniquely determined by the maps on stalks, we see:

$$f: \mathcal{M} \to \mathcal{N} \text{ is epi} \iff f_x: \mathcal{M}_x \longrightarrow \mathcal{N}_x \text{ is surjective } \forall x \in X.$$

Cokernels in the category of sheaves of modules

The cokernel of a morphism $f: M \to N$ is given by the sheafification of the presheaf

$$U \mapsto \operatorname{Coker}(M(U) \xrightarrow{f(U)} N(U))$$

Explicitly, $C := \operatorname{coker}(f)$ is given as

$$C(U) := \{(\gamma_x) \in \prod_{x \in U} \operatorname{Coker}(f_x : M_x \to N_x) \mid \text{coherence condition}\}$$

where the condition is that for every $x \in U$ there is some open nbhd V of x and an element $n \in N(V)$ s.t. for all $y \in U \cap V$, γ_y is the image of n under

$$N(V) \to N_y \to \operatorname{Coker}(f_y : M_y \to N_y)$$

Note that the stalk C_x of C is isomorphic to $\operatorname{Coker}(M_x \xrightarrow{f_x} N_x)$ by sending $(\gamma_x)/_\sim$ to γ_x.

We have a morphism $N \to C$ by sending $n \in N(U)$ to

$$(\text{image of } n \text{ under } N(U) \to N_x \to \operatorname{Coker}(M_x \xrightarrow{f} N_x))_{x \in U}$$

Since it is surjective on stalks, $N \to C$ is an epimorphism.

To see that $N \to C$ satisfies the universal property of a cokernel, let $\tau : N \to T$ be a morphism s.t. $\tau \circ f = 0$.

We define $\tau_1 : C(U) \to \prod_{x \in U} T_x$ in the following way:

for $\nu \in C(U)$, select for all $x \in U$ an $n \in N_x$ whose image in $\operatorname{coker}(M_x \xrightarrow{f_x} N_x)$ is ν_x, and set $\tau_1(\nu)_x = \tau_x(n) \in T_x$ which is independent of the choice of n.

We can show that $\tau_1(\nu)$ lands in $\operatorname{Sheaf}(T)(U) \subseteq \prod_{x \in U} T_x$, and since $T \cong \operatorname{Sheaf}(T)$, we get a morphism

$$\tau_2 : C \to T.$$

We have $\tau = \tau_2 \circ (N \to C)$, showing that $N \to C$ is indeed a cokernel of f.

Epimorphisms in the category of sheaves of modules

In any additive category, a morphism $f: M \to N$ is an epimorphism iff $\operatorname{coker}(f) = 0$.

By our previous construction of cokernels, we see

$$M \xrightarrow{f} N \text{ is epi} \iff M_x \to N_x \text{ is epi for all } x$$

$$\iff \Big(\text{For all } U \text{ and } n \in N(U) \text{ there is } U = \bigcup_{\lambda \in \Lambda} U_\lambda,\ m_\lambda \in M(U_\lambda) \text{ such that } n|_{U_\lambda} = f(m_\lambda)\Big)$$

but this does <u>not</u> imply surjectivity of $M(U) \to N(U)$.

Abeliannes of the category of sheaves of modules

If $f: M \to N$ is mono and epi, it is isomorphism on stalks, so it is an isomorphism.

If $f: M \to N$ is mono with cokernel $N \to C$, then

$$\ker(N \to C)_x = \ker(N_x \to C_x)$$
$$= \ker(N_x \to \operatorname{Coker}(M_x \xrightarrow{i_x} N_x))$$
$$\cong M_x$$

as $i_x: M_x \to N_x$ is injective.

Hence $M \to \ker(N \to C)$ induces an isomorphism on stalks and thus is iso. So monomorphisms are effective. Similar for epimorphisms.

Thus the category of sheaves of modules is Abelian.

Remark 1.1.2: Note that open subsets of the form $\operatorname{Spec}(R) \setminus V(f) \cong \operatorname{Spec}(R_f)$ form a topology base of $\operatorname{Spec} R$.

The saturation of $f^{\mathbb{N}}$ depends only on $\operatorname{Spec} R \setminus V(f)$, so that M_f depends, up to canonical isomorphism, only on $\operatorname{Spec} R \setminus V(f)$. Here, M is an R-module.

One then defines a sheaf of modules $\tilde{M}$ as the sheafification of

$$\operatorname{Spec}(R_f) \longmapsto M_f$$

Explicitly,

$$\tilde{M}(U) = \{(m_p) \in \prod_{p \in U} M_p \mid m_p \text{ locally comes from some } \mu \in M_f\}$$

A sheaf of modules on $\operatorname{Spec} R$ is called quasi-coherent (q.c.) if it is isomorphic to some $\tilde{M}$.

Recall when a prescheme is called quasi-compact and quasi-separated.

Also recall the equivalent conditions for an $\mathcal{O}_X$-module over a prescheme X to be quasi-coherent. We will add another equivalent condition,

(d) When $U \subseteq X$ is affine, the canonical morphism

$$(2) \quad M(U)_{P_x} \longrightarrow M_x$$

is an isomorphism for all $x \in U$, where

$$P_x := \{f \in \mathcal{O}_X(U) \mid x \in V(f)\}$$

Recall the adjunction

$$\operatorname{Hom}_{\mathcal{O}_X}(\tilde{M}, N) \xrightarrow{\cong} \operatorname{Hom}_R(M, N(X))$$

So when (2) is an isomorphism ($X = U = \operatorname{Spec} R$), it follows that the canonical ~~iso~~ morphism $\widetilde{M(X)} \longrightarrow M$ corresponding to $id : M(X) \longrightarrow M(X)$ is an isomorphism on stalks and thus an isomorphism. This shows the equivalence of d) with the other points.

Recall that a left adjoint functor L preserves colimits and right adjoint functor R preserves limits. In particular, if $\mathcal{A}$ and $\mathcal{B}$ are both additive, then L and R both preserve 0 and are compatible with finite direct sums.

The same happens with kernels. Explicitly:

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{A}}(T, R(\ker(f : X \to Y))) &\cong \operatorname{Hom}_{\mathcal{B}}(LT, \ker(f : X \to Y)) \\
&\cong \ker(\operatorname{Hom}_{\mathcal{B}}(LT, X) \xrightarrow{f\circ -} \operatorname{Hom}_{\mathcal{B}}(LT, Y)) \\
&\cong \ker(\operatorname{Hom}_{\mathcal{A}}(T, RX) \xrightarrow{Rf\circ -} \operatorname{Hom}_{\mathcal{A}}(T, RY)) \\
&\cong \operatorname{Hom}_{\mathcal{A}}(T, \ker(RX \xrightarrow{Rf} RY))
\end{aligned}$$

so R preserves kernels and similarly L preserves cokernels.

Let $X \cong \operatorname{Spec} R$ be an affine prescheme. Define

$$\mathcal{A} = R\text{-modules}$$

$$\mathcal{B} = \mathcal{O}_X\text{-modules}$$

$$L : \mathcal{A} \longrightarrow \mathcal{B} \; : \; M \longmapsto \tilde{M}$$

$$R : \mathcal{B} \longrightarrow \mathcal{A} \; : \; M \longmapsto M(X)$$

Then L is left adjoint to R

Recall when a ~~···~~ morphism $f : X \longrightarrow Y$ of preschemes is q.c. or q.s. ($f^{-1}(\text{q.c.}) = \text{q.c.}$, $f^{-1}(\text{q.s.}) = \text{q.s.}$)

<u>Proposition 1.1.2</u>: If $f : X \longrightarrow Y$ is q.s. and q.c. and $\mathcal{M} \in \underline{\mathcal{Q}c}(X)$ (the category of q.c. $\mathcal{O}_X$-modules), then $f_* \mathcal{M} \in \underline{\mathcal{Q}c}(Y)$.

<u>Proposition 1.1.3</u>: a) The class $\underline{\mathcal{Q}c}(X)$ is closed under taking kernels and cokernels of morphisms and under taking (finite) direct sums.

b) If $\mathcal{M}$ is a q.c. $\mathcal{O}_X$-module, $U \subseteq X$ open, then $\mathcal{M}|_U \in \underline{\mathcal{Q}c}(U)$

Proposition 1.1.4: If X is a prescheme, then we have a bijection

$$\{\text{closed subpreschemes of } X\} \xrightarrow{\cong} \{\text{q.c. ideals in } \mathcal{O}_X\}$$
$$(f: Y \longrightarrow X) \longmapsto \ker(\mathcal{O}_X \xrightarrow{f^*} f_* \mathcal{O}_Y)$$

Lemma 1.1.1: For a q.c. $\mathcal{O}_X$-module M on a prescheme X, t.f.a.e:

a) For any open $U \subseteq X$, $M(U)$ is a f.g. $\mathcal{O}_X(U)$-module.

b) There is a covering of X with such U.

For proving this we used a useful lemma:

Sublemma A: If E is a property of affine open subsets of a prescheme X s.t.

- If $E(U)$, then $E(U \setminus V(f))$
- If $E(U \setminus V(f_i))$ for all i, and $\bigcap_{i=1}^{n} V(f_i) = \emptyset$, then $E(U)$.

Then the following are equivalent:

a) If U is open affine, then $E(U)$ holds.

b) There is an open covering of U s.t. $E(U)$ holds.

Definition 1.1.3: • We call a q.c. $\mathcal{O}_X$-module M <u>locally finitely generated</u> if it satisfies the equivalent conditions of lemma 1.1.1.

• When X is a locally Noetherian prescheme, an $\mathcal{O}_X$-module is <u>coherent</u> if it is q.c. and locally fintely generated.

Remark: There is a general definition of "coherent sheaves of modules" on arbitrary ringed spaces, which in the case of locally Noetherian preschemes is equivalent to the above.

<u>Addendum to proposition 1.1.3 and its proof</u>

As $M \longmapsto \tilde{M}$ from <u>R-Mod</u> to <u>$\mathcal{O}_X$-Mod</u> is left adjoint to the global section functor, it preserves cokernels. It follows that

Corollary 1.1.1: Let Y be any prescheme, $f: M \longrightarrow N$ a morphism of q.c. $\mathcal{O}_Y$-modules and $X \subseteq Y$ affine open.

Then

$$\ker(M(X) \xrightarrow{f(X)} N(X)) \cong (\ker(f))(X)$$

and

$$\operatorname{coker}(M(X) \xrightarrow{f(X)} N(X)) \cong (\operatorname{coker}(f))(X).$$

Proof: The first assertion holds for arbitrary open $X \subseteq Y$ by our explicit description of $\ker(f)$. For the second one we may assume $X = Y = \operatorname{Spec}(R)$, as restricting preserves colimits. (or only cokernels?).

Then for $M = M(X)$, $N = N(X)$, we have a commutative diagram

$$\begin{tikzcd} M \arrow[r, "f"] & N \arrow[r] & \mathrm{coker}(f) \arrow[r] & 0 \\ \tilde{M} \arrow[u, "\cong"] \arrow[r] & \tilde{N} \arrow[u, "\cong"] \arrow[r] & \widetilde{\mathrm{coker}(M \to N)} \arrow[u] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

and the assertion follows.

<u>Corollary 1.1.2</u>: Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of q.c. $\mathcal{O}_X$-modules on a prescheme X and $U \subseteq X$ affine. Then

$$0 \to M'(U) \to M(U) \to M''(U) \to 0$$

is exact.

<u>Remark</u>:

We define $\mathrm{im}(\mathcal{A} \xrightarrow{f} \mathcal{B}) = \ker(\mathcal{B} \to \mathrm{coker}(f))$.

For sheaves of modules

$$(\mathrm{Im}(\mathcal{A} \xrightarrow{f} \mathcal{B}))(U) = \{ b \in \mathcal{B}(U) \mid b_x \in \mathrm{Im}(f : \mathcal{A}_x \to \mathcal{B}_x) \ \forall x \}$$

$$= \{ b \in \mathcal{B}(U) \mid \text{Can cover } U = \bigcup U_i \text{ s.t. } b|_{U_i} \in \mathrm{Im}(\mathcal{A}(U_i) \to \mathcal{B}(U_i)$$

A sequence $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is <u>exact</u> iff $\mathrm{im}(\alpha) = \ker(\beta)$.

<u>Remark</u>: In a ses. $0 \to M' \to M \to M'' \to 0$, if two of the three $\mathcal{O}_X$-modules are q.c. then the third is as well.

Our plan is to associate to any q.c. $\mathcal{O}_X$-module M on a scheme X the <u>cohomology groups</u> $\cancel{H^i(M)}$, $H^i(X, M)$ s.t.

- $H^0(M, X) \cong M(X)$
- When $0 \to M' \to M \to M'' \to 0$ is exact, we have a canonical long exact cohomology sequence:

$$\begin{tikzcd} 0 \arrow[r] & H^0(X,M') \arrow[r] & H^0(X,M) \arrow[r] & H^0(X,M'') \arrow[dll, rounded corners] \\ & H^1(X,M') \arrow[r] & H^1(X,M) \arrow[r] & H^1(X,M'') \arrow[r] & \cdots \end{tikzcd}$$

1.2 Čech-cohomology

Motivation: if $\mathcal{U}$ is a cover of $X = \bigcup_{i \in I} U_i$, we have a l.e.s.

$$\check{C}^*(\mathcal{U}, M) : \prod_{i \in I} M(U_i) \to \prod_{(i,j)} M(U_{ij}) \longrightarrow \prod_{(i,j,k)} M(U_{ijk}) \to$$

$$(m_i) \longmapsto (m_i|_{U_{ij}} - m_j|_{U_{ij}})$$

Note that the kernel of this first map is isomorphic to $M(X)$ by the sheaf axiom.

<u>Definition 1.2.1</u>: Let $\mathfrak{U}: X = \bigcup_{i\in I} U_i$ be an open cover of a topological space X (e.g. a prescheme) and let $\mathcal{M}$ be a presheaf of abelian groups (e.g. of an $\mathcal{O}_X$-module) on X. Let always

(1) $$U_{i_0,\dots,i_n} := \bigcap_{k=0}^{n} U_{i_k}$$

The <u>Čech complex</u> $\check{C}^*(\mathfrak{U},\mathcal{M})$ is the cochain complex defined as

$$\check{C}^k(\mathfrak{U},\mathcal{M}) := \prod_{(i_0,\dots,i_k)\in I^{k+1}} \mathcal{M}(U_{i_0,\dots,i_k})$$

Let elements of $\check{C}^k(\mathfrak{U},\mathcal{M})$ be denoted $(\varphi_{i_0\dots i_k})_{(i_0\dots i_k)\in I^{k+1}}$.
Then let $\check{d}: \check{C}^k(\mathfrak{U},\mathcal{M}) \longrightarrow \check{C}^{k+1}(\mathfrak{U},\mathcal{M})$ be defined by

$$(\check{d}\varphi)_{(i_0,\dots,i_{k+1})} = \sum_{j=0}^{k+1} (-1)^j \underbrace{\varphi_{i_0\dots\hat{i_j}\dots i_{k+1}}|_{U_{i_0\dots i_{k+1}}}}_{d_j\varphi}.$$

For instance:

$k=0$: $$(\check{d}\varphi)_{ij} = \varphi_j|_{U_{ij}} - \varphi_i|_{U_{ij}}$$

$k=1$: $$(\check{d}\varphi)_{ijk} = \varphi_{ij}|_{U_{ijk}} + \varphi_{jk}|_{U_{ijk}} - \varphi_{ik}|_{U_{ijk}}$$

etc.

Let $\check{H}^i(\mathfrak{U},\mathcal{M}) := H^i(\check{C}^*(\mathfrak{U},\mathcal{M}))$

It is easy to see that each term in $\check{d}^2\varphi$ appears twice with opposite sign and thus that $\check{d}^2\varphi = 0$.

<u>Remark</u>: Our program is to show that $\check{H}^*(\mathfrak{U},\mathcal{M})$ is up to canonical isomorphism independent of $\mathfrak{U}$ if $\mathfrak{U}$ is an affine open covering of the <u>scheme</u> X and $\mathcal{M}\in Qc(X)$, and that it has the desired properties.

<u>Remark 1.2.1</u> For instance, the cohomology of $\mathbb{P}^1_R = \mathrm{Proj}(R[X_0,X_1])$ can be calculated using the affine open covering

$$U_i = \mathbb{P}^1_R \setminus V(X_i) \cong \mathrm{Spec}(R[X_0,X_1][X_i^{-1}])_0$$
$$\cong \mathrm{Spec}(R[\Xi_i]) \qquad \Xi_i := \begin{cases} X_1/X_0 & i=0 \\ X_0/X_1 & i=1 \end{cases}$$

Unfortunately, calculations are difficult.

<u>Remark 1.2.2</u>: Let $\check{C}^{a\,k}(\mathfrak{U},\mathcal{M}) \subseteq \check{C}^k(\mathfrak{U},\mathcal{M})$ be the subgroup containing all $\varphi\in\check{C}^k(\mathfrak{U},\mathcal{M})$ such that for all $\pi\in S_{k+1}$

$$\varphi_{i_{\pi(0)},\dots,i_{\pi(k)}} = \mathrm{sgn}(\pi)\,\varphi_{i_0\dots i_k} \in \mathcal{M}(U_{i_0\dots i_k} = U_{i_{\pi(0)},\dots,i_{\pi(k)}})$$

and

$$\varphi_{i_0\dots i_k} = 0 \quad \text{if} \quad i_m = i_n \quad \text{for some} \quad m\neq n.$$

We want to show that ${}^{a}\check{C}^*(\mathfrak{U},M) \subseteq \check{C}^*(\mathfrak{U},M)$ is a subcomplex. For each $0 \le \ell \le k-1$ define $s_\ell : \check{C}^k \longrightarrow \check{C}^{k-1}$ and $t_\ell : \check{C}^k \longrightarrow \check{C}^k$ by

$$(s_\ell \psi)_{i_0 \dots i_{k-1}} = \psi_{i_0 \dots i_\ell i_\ell \dots i_{k-1}}$$

$$(t_\ell \psi)_{i_0 \dots i_k} = \psi_{i_0 \dots i_{\ell+1} i_\ell \dots i_k}$$

Clearly for $\psi \in {}^{a}\check{C}^k$ we have

$$s_\ell \psi = 0$$
$$t_\ell \psi = -\psi$$

We also have

$$(3) \quad s_\ell d_i = \begin{cases} d_i s_{\ell-1} & i < \ell \\ \mathrm{Id} & i = \ell \text{ or } i = \ell+1 \\ d_{i-1} s_\ell & i > \ell+1 \end{cases}$$

and

$$(4) \quad t_\ell d_j = \begin{cases} d_j t_\ell & \ell < j-1 \\ d_\ell & \ell = j-1 \\ d_{\ell+1} & \ell = j \\ d_j t_{\ell+1} & \ell > j \end{cases}$$

This implies that ${}^{a}\check{C}^*$ is a subcomplex e.g.

$$\begin{aligned} t_\ell \check{d} \psi &= \sum_{j=0}^{\ell-1} (-1)^j t_\ell d_j \psi + (-1)^\ell t_\ell d_\ell \psi + (-1)^{\ell+1} t_\ell d_{j+1} + \sum_{j=\ell+2}^{k} (-1)^j t_\ell d_j \psi \\ &= \sum_{j=0}^{\ell-1} (-1)^j d_j t_\ell \psi + (-1)^\ell d_{\ell+1} \psi + (-1)^{\ell+1} d_\ell \psi + \sum_{j=\ell+2}^{k} (-1)^j d_j t_{\ell+1} \psi \\ &= -\sum_{j=0}^{\ell-1} (-1)^j d_j \psi - (-1)^\ell d_\ell \psi - (-1)^{\ell+1} d_{\ell+1} \psi - \sum_{j=\ell+2}^{k} (-1)^j d_j \psi \\ &= -\check{d}\psi = \check{d}(t_\ell \psi) \end{aligned}$$

and similarly $s_\ell \check{d} \psi = 0$ when $s_m \psi = 0$.

Remark 1.2.3: A cosimplicial object of a category is a sequence of objects (X^n) with morphisms $d_j : X^k \longrightarrow X^{k+1}$ when $0 \le j \le k+1$ satisfying (2) and $s_j : X^k \longrightarrow X^{k-1}$ where $0 \le j \le k$ satisfying a version of (2) together with (3). The s_j are called codegeneragy maps.

There is a Dold-Puppe correspondence between cochain complexes concentrated in non-negative degrees and cosimplicial objects of an Abelian category.

Example 1.2.1: a) By the sheaf axiom, for $X = \bigcup_{i \in I} U_i$,

$$\begin{aligned} M(X) &\cong \ker(\check{C}^0(\mathfrak{U},M) \to \check{C}^1(\mathfrak{U},M)) \\ &\cong \ker({}^{a}\check{C}^0(\mathfrak{U},M) \to {}^{a}\check{C}^1(\mathfrak{U},M)) \\ &= {}^{a}\check{Z}^0(\mathfrak{U},M) \\ &= {}^{a}\check{H}^0(\mathfrak{U},M) \qquad (= \check{H}^0(\mathfrak{U},M)) \end{aligned}$$

b) For the trivial covering $\mathfrak{U}: X = X$.

$${}^a\check{C}^*(\mathfrak{U}, M): \quad M(X) \longrightarrow 0 \longrightarrow 0 \longrightarrow \dots$$

$$\check{C}^*(\mathfrak{U}, M): M(X) \xrightarrow{0} M(X) \xrightarrow{id} M(X) \xrightarrow{0} \dots$$

so

$${}^a\check{H}^k(\mathfrak{U}, M) = \check{H}^k(\mathfrak{U}, M) = \begin{cases} M(X) & k=0 \\ 0 & k>0 \end{cases}$$

c) We have

$$\check{C}^*(\mathfrak{U}, A \oplus B) = \check{C}(\mathfrak{U}, A) \oplus \check{C}(\mathfrak{U}, B)$$

and similarly for ${}^a\check{C}^*$

In fact

$$\check{C}^*(\mathfrak{U}, \prod_{i\in I} M_i) \cong \prod_{i\in I} \check{C}^*(\mathfrak{U}, M_i)$$

and similarly for ${}^a\check{C}^*$.

By corollary 1.1.2 and proposition 1.1.1, we have for a short exact sequence $0 \to M' \to M \to M'' \to 0$ in $Qc(X)$ in a scheme X and an <u>affine</u> open cover $\mathfrak{U}$ the two short exact sequences

$$0 \longrightarrow \check{C}^*(\mathfrak{U}, M') \longrightarrow \check{C}^*(\mathfrak{U}, M) \longrightarrow \check{C}^*(\mathfrak{U}, M'') \longrightarrow 0$$

$$0 \longrightarrow {}^a\check{C}^*(\mathfrak{U}, M') \longrightarrow {}^a\check{C}^*(\mathfrak{U}, M) \longrightarrow {}^a\check{C}^*(\mathfrak{U}, M'') \longrightarrow 0$$

<u>Proposition 1.2.1</u>: If $\mathfrak{U}$ is an affine cover of a scheme X and $M \in Qc(X)$, then we have a long exact cohomology sequence

$$\begin{array}{ccccccc} & M'(X) & & M(X) & & M''(X) & \\ & \wr\wr & & \wr\wr & & \wr\wr & \\ 0 \longrightarrow & \check{H}^0(\mathfrak{U}, M') & \longrightarrow & \check{H}^0(\mathfrak{U}, M) & \longrightarrow & \check{H}^0(\mathfrak{U}, M'') & \xrightarrow{d} \\ \longrightarrow & \check{H}^1(\mathfrak{U}, M') & \longrightarrow & \check{H}^1(\mathfrak{U}, M) & \longrightarrow & \check{H}^1(\mathfrak{U}, M'') & \xrightarrow{d} \dots \end{array}$$

The same holds for ${}^a\check{H}^*(\mathfrak{U}, -)$ $\square$

<u>Remark 1.2.4</u>: For preschemes the situation is more difficult.

<u>Definition 1.2.2</u>: A cover $\mathfrak{V}: X = \bigcup_{j\in J} V_j$ of X is called a <u>refinement</u> of $\mathfrak{U}$ if there is a map $\nu: J \to I$ s.t. $V_j \subseteq U_{\nu(j)}$ (i.e. by axiom of choice, if every V_j is contained in some U_i)

The map $\nu: J \to I$ will be called a refinement map (in this lecture.)

It defines a morphism

$$\nu^*: \check{C}^*(\mathfrak{U}, M) \longrightarrow \check{C}^*(\mathfrak{V}, M)$$

$$\psi \longmapsto \nu^*(\psi)$$

$$\nu^*(\psi)_{j_0\dots j_k} := \psi_{\nu(j_0)\dots\nu(j_k)}|_{V_{j_0\dots j_k}}$$

which maps ${}^a\check{C}^*$ to ${}^a\check{C}^*$, as ν^* commutes with d_i, s_i and t_i.

Lemma 1.2.1 a) The refinement $\mathcal{W}$ of a refinement $\mathcal{V}$ of $\mathcal{U}$ is a refinement of $\mathcal{U}$ and the composition of two refinement maps $K \xrightarrow{w} J \xrightarrow{v} I$ is a refinement map and $(vw)^* = w^* v^*$.
Moreover, $id_{\mathbb{I}}$ is a refinement map and $Id_{\mathbb{I}}^* = Id_{\check{C}^*(\mathcal{U}.)}$.
b) Two arbitrary open covers $\mathcal{U}$ and $\mathcal{V}$ always have a common refinement (e.g. $X = \bigcup_{(i,j) \in I \times J} U_i \cap V_j$). When X is a prescheme, this common refinement may be chosen to be a cover by open affine subsets.
c) If $J \xrightarrow{v, \hat{v}} I$ are two refinement maps for $\mathcal{V}$, then $\check{C}^*(\mathcal{U}, M) \overset{v^*}{\underset{\hat{v}^*}{\rightrightarrows}} \check{C}(\mathcal{V}, M)$ induce the same morphism $\check{H}^*(\mathcal{U}, M) \longrightarrow \check{H}^*(\mathcal{V}, M)$ on cohomology.

Proof: For c), define $h: \check{C}^k(\mathcal{U}, M) \longrightarrow \check{C}^{k+1}(\mathcal{U}, M)$ by $h = \sum_{\ell=0}^{k-1} h_\ell$.
where

$$(h_\ell \psi)_{j_0 \ldots j_{k-1}} = \psi_{v(j_0), \ldots, v(j_\ell), \hat{v}(j_\ell), \hat{v}(j_{\ell+1}), \ldots, \hat{v}(j_{k-1})} \Big|_{V_{j_0 \ldots j_{k-1}}}$$

Then

$$h_\ell d_j = \begin{cases} d_j h_{\ell-1} & 0 \le j < \ell \\ h_{\ell-1} d_{j+1} & 0 < j = \ell \\ \hat{v}^* & i = \ell = 0 \\ h_{\ell+1} d_{j-1} & j = \ell+1 < k \\ v^* & j = \ell+1 = k \\ d_{j+1} h_\ell & j > \ell+1 \end{cases}$$

implying that

$$h\check{d} + \check{d}h = v^* - \hat{v}^*$$

so v^* and $\hat{v}^*$ are chain homotopic.

Corollary 1.2.1 a) When $\mathcal{V}$ is a refinement of $\mathcal{U}$, we have a canonical morphism

$$T_{\mathcal{U}\mathcal{V}}: \check{H}^*(\mathcal{U}, M) \longrightarrow \check{H}^*(\mathcal{V}, M).$$

We have $T_{\mathcal{U}\mathcal{U}} = Id$ and $T_{\mathcal{U}\mathcal{W}} = T_{\mathcal{V}\mathcal{W}} \circ T_{\mathcal{U}\mathcal{V}}$
b) When $\mathcal{U}$ is a refinement of $\mathcal{V}$ and $\mathcal{V}$ is a refinement of $\mathcal{U}$, then $T_{\mathcal{U}\mathcal{V}}$ is an isomorphism (with inverse $T_{\mathcal{V}\mathcal{U}}$)
c) If there is $i_* \in I$ s.t. $U_{i_*} = X$, then $\check{H}^k(\mathcal{U}, M) = 0$ for $k > 0$.

Remark 1.2.5: In general, the homotopy used for the proof of lemma 1.2.1 will not preserve the subcomplex $\check{\check{C}}^* \subseteq \check{C}^*$. However, corollary 1.2.1 c) can be obtained by using the chain contraction

$$(h\psi)_{i_0 \ldots i_k} = \psi_{i_0 \ldots i_k, i_*}$$

of $\check{C}(\mathcal{U}, M)$ in positive degrees, preserving $\check{\check{C}}^* \subseteq \check{C}^*$. For our purposes, this is sufficient.

<u>Proposition 1.2.2</u>: a) Let $\mathcal{U}$ and $\mathcal{V}$ be affine coverings of a q.c. scheme X, where $\mathcal{V}$ is a refinement of $\mathcal{U}$. Then for every q.c. $\mathcal{O}_X$-module $\mathcal{M}$ on X, the homomorphism

$$\mathrm{Tur}: \check{H}^*(\mathcal{U}, \mathcal{M}) \longrightarrow \check{H}^*(\mathcal{V}, \mathcal{M})$$

is an isomorphism.

b) In the above situation,

$$^a\check{H}^*(\mathcal{U}, \mathcal{M}) \longrightarrow \check{H}^*(\mathcal{U}, \mathcal{M})$$

is an isomorphism.

c) If X is affine and $i > 0$, then

$$\check{H}^i(\mathcal{U}, \mathcal{M}) = {}^a\check{H}^i(\mathcal{U}, \mathcal{M}) = 0$$

<u>Lemma 1.2.2</u>: For an open covering $\mathcal{U}: X = \bigcup_{i \in I} U_i$ of a topological space X and a continuous map $f: Y \to X$, let $f^{-1}\mathcal{U}$ be the covering

$$Y = \bigcup_{i \in I} f^{-1}(U_i)$$

Let $\mathcal{F}$ be a sheaf of Abelian groups on Y.

a) $\check{C}^*(f^{-1}\mathcal{U}, \mathcal{F}) \cong \check{C}^*(\mathcal{U}, f_*\mathcal{F})$

and this isomorphism restricts to an iso

$$^a\check{C}^*(f^{-1}\mathcal{U}, \mathcal{F}) \cong {}^a\check{C}^*(\mathcal{U}, f_*\mathcal{F})$$

b) If the image of f is contained in one of the open subsets U_i, then

$$\check{H}^i(\mathcal{U}, f_*\mathcal{F}) \cong {}^a\check{H}^i(\mathcal{U}, f_*\mathcal{F}) = 0 \quad \text{when } i > 0.$$

<u>Proof</u>: a) Let $\mathcal{V} := f^{-1}\mathcal{U}$, $V_i = f^{-1}U_i$, then

$$V_{i_0 \dots i_k} = f^{-1}(U_{i_0 \dots i_k})$$

so

$$(f_*\mathcal{F})(U_{i_0 \dots i_k}) = \mathcal{F}(V_{i_0 \dots i_k})$$

and thus

$$\check{C}^*(\mathcal{U}, f_*\mathcal{F}) = \check{C}^*(\mathcal{V}, \mathcal{F})$$

b) Follows from a), using corollary 1.2.1c) and remark 1.2.5.

<u>Proof of proposition 1.2.2</u> a) We fix the affine coverings $\mathcal{U}$ and $\mathcal{V}$ and consider the following condition on $\mathcal{M}$:

$$A_i(\mathcal{M}): \boxed{\check{H}^j(\mathcal{U}, \mathcal{M}) \xrightarrow{\mathrm{Tur}} \check{H}^j(\mathcal{V}, \mathcal{M}) \text{ is an isomorphism for } j < i \text{ and injective for } j = i}$$

We first claim that for a s.e.s. $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ of q.c. $\mathcal{O}_X$-modules, one has the implication

(+) $\quad (A_{i+1}(\mathcal{B}) \text{ and } A_i(\mathcal{C})) \Longrightarrow A_{i+1}(\mathcal{A})$

In fact we have the following commutative diagram

$$\begin{tikzcd}
\to \check{H}^{j-1}(\mathcal{U},\mathcal{B}) \arrow[r] \arrow[d] & \check{H}^{j-1}(\mathcal{U},\mathcal{C}) \arrow[r] \arrow[d] & \check{H}^{j}(\mathcal{U},\mathcal{A}) \arrow[r] \arrow[d] & \check{H}^{j}(\mathcal{U},\mathcal{B}) \arrow[r] \arrow[d] & \check{H}^{j}(\mathcal{U},\mathcal{C}) \to \arrow[d] \\
\to \check{H}^{j-1}(\mathcal{V},\mathcal{B}) \arrow[r] & \check{H}^{j-1}(\mathcal{V},\mathcal{C}) \arrow[r] & \check{H}^{j}(\mathcal{V},\mathcal{A}) \arrow[r] & \check{H}^{j}(\mathcal{V},\mathcal{B}) \arrow[r] & \check{H}^{j}(\mathcal{V},\mathcal{C}) \to
\end{tikzcd}$$

proving the bijectivity ($j<i$) or injectivity ($i=j$) of $\tau_{\mathcal{U}\mathcal{V}}$.

We now also claim that for any q.c. $\mathcal{O}_X$-module $\mathcal{M}$, there is a q.c. $\mathcal{O}_X$-module $\mathcal{N}$ with $A_i(\mathcal{N})$ for all i and a monomorphism

$$\mathcal{M} \longrightarrow \mathcal{N}$$

To show this let $j_W : W \longrightarrow X$ be the embedding of any affine open subset contained in one of the open subsets forming the covering $\mathcal{V}$.

As X is a scheme the morphism j_W is affine (proposition 1.1.1), and hence q.c. Being an open embedding, it is also q.s. By proposition 1.1.2 $(j_W)_*(\mathcal{M}|_W)$ is a q.c. $\mathcal{O}_X$-module. Since X is q.c., there are finitely many $(W_i)_{i=1}^n$ which are affine open subsets of X covering X and contained in an element of $\mathcal{V}$ (e.g. a finite subcovering of $\mathcal{V}$).

Then

$$\mathcal{N} := \bigoplus_{i=1}^n (j_{W_i})_*(\mathcal{M}|_{W_i})$$

is q.c. by proposition 1.1.3. We have morphisms

$$\begin{aligned}
\mathcal{M}(U) &\longrightarrow \mathcal{N}(U) = \bigoplus_{i=1}^n (j_{W_i})_*(\mathcal{M}|_{W_i})(U) = \bigoplus_{i=1}^n \mathcal{M}(U \cap W_i) \\
m &\longmapsto (m|_{U\cap W_i})_{i=1}^n
\end{aligned}$$

which are injective, as $\mathcal{M}$ satisfies the sheaf axiom. We get a monomorphism $\mathcal{M} \longrightarrow \mathcal{N}$.

By example 1, $\check{H}^*(\mathcal{U},\mathcal{N}) \cong \bigoplus_{i=1}^n \check{H}^*(\mathcal{U}, (j_{W_i})_*\mathcal{N}|_{W_i})$ and similarly for $\mathcal{V}$, so $\mathcal{N}$ inherits properties A_j from its summands.

This finishes the second claim.

Now we show by induction that $A_i(\mathcal{M})$ holds for all $\mathcal{M}$.

For $i=0$, this is clear as $\check{H}^0(\mathcal{U},\mathcal{M}) = \mathcal{M}(X) = \check{H}^0(\mathcal{V},\mathcal{M})$.

For $i>0$, we choose a monomorphism $\mathcal{M} \overset{\iota}{\hookrightarrow} \mathcal{N}$ and get a s.e.s.

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{N} \longrightarrow \underbrace{\operatorname{coker}(\iota)}_{\text{(q.c. by prop 1.1.3)}} \longrightarrow 0.$$

Then by (+), $A_i(\mathcal{N})$ and $A_{i-1}(\operatorname{coker}(\iota))$ imply $A_i(\mathcal{M})$.

b) The previous proof carries over with the following modifications.

- $A_i(\mathcal{M})$: ${}^a\check{H}^j(\mathfrak{U},\mathcal{M}) \longrightarrow \check{H}^j(\mathfrak{U},\mathcal{M})$ is iso when $j<i$ and mono when $j=i$.
- $(+)$ also holds for these new A_i
- If $W \subseteq X$ is affine and contained in one of the U_i, and $\mathcal{F} \in \mathcal{Q}c(W)$, then $(j_W)_*\mathcal{F}$ satisfies all A_i.

c) Apply a) with $\mathfrak{U} = (X)$, $\mathfrak{V} =$ (any affine covering of X) to conclude that

$$\check{H}^i(\mathfrak{V},\mathcal{M}) = 0 \qquad (\text{when } i>0,\ \mathcal{M} \text{ q.c.})$$

Then one uses b) to show that ${}^a\check{H}^i(\mathfrak{V},\mathcal{M})$ also vanishes.

<u>Definition 1.2.3</u> Let X be a q.c. scheme and $\mathcal{M}$ a q.c. $\mathcal{O}_X$-module. We put

$$H^{\cdot}(X,\mathcal{M}) := \check{H}^{\cdot}(\mathfrak{U},\mathcal{M})$$

where ~~$\mathfrak{X}$~~ $\mathfrak{U}$ is the covering of X by all affine open subsets.

<u>Theorem 1</u> : Let X be a q.c. scheme and $\mathcal{M}$ a q.c. $\mathcal{O}_X$-module.

a) If $\mathfrak{V}$ is any affine covering of X, then

$$H^*(X,\mathcal{M}) \xrightarrow{\ \mathfrak{I}_{\mathfrak{U}}\ } \check{H}^*(\mathfrak{U},\mathcal{M}) \longleftarrow {}^a\check{H}^*(\mathfrak{V},\mathcal{M})$$

are isomorphisms compatible with the isomorphisms

$$\check{H}^*(\mathfrak{V},\mathcal{M}) \longrightarrow \check{H}^*(\mathfrak{W},\mathcal{M})$$

for an affine refinement of $\mathfrak{V}$.

(b) When X is affine, $H^i(X,\mathcal{M}) = 0$ when $i>0$

c) There is a canonical isomorphism

$$H^0(X,\mathcal{M}) \xrightarrow{\ \sim\ } \mathcal{M}(X).$$

d) When $0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}'' \longrightarrow 0$ is a s.e.s of q.c $\mathcal{O}_X$-modules one has a long exact cohom seq.

$$0 \longrightarrow H^0(X,\mathcal{M}') \longrightarrow H^0(X,\mathcal{M}) \longrightarrow H^0(\mathcal{M}'') \longrightarrow H^1(X,\mathcal{M}') \longrightarrow \cdots$$

<u>Corollary 1.2.2</u> : Let $0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}'' \longrightarrow 0$ be a short exact sequence of $\mathcal{O}_X$-modules, where X is any prescheme. If two of the three $\mathcal{O}_X$-modules $\mathcal{M}', \mathcal{M}, \mathcal{M}''$ are q.c., then so is the third and for any affine open $U \subseteq X$ the sequence

$$0 \longrightarrow \mathcal{M}'(U) \longrightarrow \mathcal{M}(U) \longrightarrow \mathcal{M}''(U) \longrightarrow 0$$

is exact.

Proof For M' and M'' we saw this in prop 1.1.3 and cor 1.1.1 about kernels and cokernels. We may thus assume that M' and M'' are q.c.
For showing exactness we may do this locally and assume $\mathcal{U} = X = \operatorname{Spec} R$. Only surjectivity of $M(\mathcal{U}) \longrightarrow M''(\mathcal{U})$ needs to be shown. Let $m \in M''(\mathcal{U})$. As $M \to M''$ is epi, there is a covering of X by affine open subsets V_i such that $m''|_{V_i}$ is in the image of $M(V_i) \longrightarrow M''(V_i)$. As X is q.c., we may assume this to be a finite cover. Let $m_i \in M(V_i)$ be chosen whose image in $M''(V_i)$ is $m''|_{V_i}$. Then the images of $m_i|_{V_{ij}} - m_j|_{V_{ij}}$ vanishes, so there are $m_{ij}' \in M'(V_{ij})$ whose image under $M'(V_{ij}) \longrightarrow M(V_{ij})$ is $m_i|_{V_{ij}} - m_j|_{V_{ij}}$. This means that the image under $\check{C}^1(\mathcal{V}, M') \longrightarrow \check{C}^1(\mathcal{V}, M)$ of $\mu' \in \check{C}^1(\mathcal{V}, M')$ (given by the m_{ij}') is equal to $d\mu$ (μ given by the m_i). The image under the injective map $\check{C}^2(\mathcal{V}, M') \longrightarrow \check{C}^2(\mathcal{V}, M)$ of $d\mu'$ is thus $\check{d}\check{d}\mu = 0$, so $\check{d}\mu' = 0$ and by the vanishing of $\check{H}^1(\mathcal{V}, M')$ (as X affine by theorem 1) there are $(\tilde{\mu}_i)_{i=1}^n$, $\tilde{\mu}_i \in M'(V_i)$ s.t.

$$\tilde{\mu}_i|_{V_{ij}} - \tilde{\mu}_j|_{V_{ij}} = \mu'_{ij}.$$

For $\hat{m}_i = m_i -$ (image of $\tilde{\mu}_i$ under $M'(V_i) \longrightarrow M(V_i)$), this implies $\hat{m}_i|_{V_{ij}} = \hat{m}_j|_{V_{ij}}$, and as M is a sheaf, there is $\hat{m} \in M(X)$ s.t. $\hat{m}|_{V_i} = \hat{m}_i$. The image of $\hat{m}$ in $M''(X)$ is m'', as its restrictions to V_i are

$$\begin{aligned}(\text{image in } M''(V_i) \text{ of } \hat{m}_i) &= (\text{image of } M''(V_i) \text{ of } \\ &\quad m_i - (\text{image of } \tilde{m}_i \text{ under } M'(V_i) \to M(V_i))) \\ &= (\text{image of } m_i \text{ in } M''(V_i)) = m''|_{V_i}.\end{aligned}$$

This proves surjectivity of $M(\mathcal{U}) \longrightarrow M''(\mathcal{U})$.
To show q.c.ness of M, we consider the exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

where $M = M(X)$, $M' = M'(X)$, $M'' = M''(X)$.
Since $(\tilde{N})_{\mathfrak{p}} \cong N_{\mathfrak{p}}$ (canonical iso), the sequence of stalks of

$$0 \longrightarrow \tilde{M}' \longrightarrow \tilde{M} \longrightarrow \tilde{M}'' \longrightarrow 0$$

is exact for arbitrary $\mathfrak{p} \in \operatorname{Spec} R$, showing the exactness of the top row in

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{M}' & \longrightarrow & \tilde{M} & \longrightarrow & \tilde{M}'' & \longrightarrow & 0 \\ & & \downarrow\cong & & \downarrow & & \downarrow\cong & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \end{array}$$

where two of the vertical arrows are isomorphisms as M' and M'' are q.c. By the five-lemma, the remaining map is an iso, showing q.c.-ness of M.

Corollary 1.2.3: Let X be a q.c. scheme.

a) If $W \xrightarrow{j=j_W} X$ is the embedding of an affine open subset W and $M \in Qc(W)$, then
$$H^p(X, j_* M) = 0 \quad \text{for all } p>0$$

b) If $W = \bigcup_{i=1}^{n} W_i$, where W_i is affine, and $M \in Qc(X)$, then
$$H^p(X, \bigoplus_{j=1}^{n} (j_{W})_* (M|_{W_i})) = 0 \quad \text{for all } p>0$$

and if $\mathcal{K}$ denotes the kernel of
$$M \longrightarrow \bigoplus_{j=1}^{n} (j_{W_i})_*(M|_{W_i})$$
$$(m \in M(U)) \longmapsto (m|_{U\cap W_i})_{i=1}^{n} \in \prod_{i=1}^{n} M(U\cap W_i) = \prod_{i=1}^{n}((j_{W_i})_* M|_{W_i})(U))$$

then $\mathcal{K}|_W = 0$.

Proof a) Let $\mathcal{U}$ be any open affine cover of X. Then by $U_i \cap W$ being affine (prop 1.1.1) we have
$$\check{H}^p(\mathcal{U}\cap W, M) \cong \check{H}^p(\mathcal{U}, j_* M) = 0$$
when $p>0$ by theorem 1.

b) Similar to the assertion in theorem 1.

Lemma 1.2.3: Let $\mathcal{U}$ be a finite open cover of the quasi-separated prescheme X by ~~the~~ quasi-compact open subsets and $f \in \mathcal{O}_X(X)$. By the universal property of localization w.r.t. f, we have morphisms
$$M(U)_f \longrightarrow M(U \setminus V(f))$$
for arbitrary $\mathcal{O}_X$-modules M and open $U \subseteq X$. Applied to $U = U_{i_0,\dots,i_k}$, this gives a morphism
$$\check{C}(\mathcal{U}, M)_f \longrightarrow \check{C}(\mathcal{U} \cap (X \setminus V(f)), M)$$
of Čech-complexes.

When M is quasi-coherent, this is an isomorphism.

Remark: The structure of an $\mathcal{O}_X(X)$-module on $\check{C}^*(\mathcal{U}, M)$ is obtained by using the structures of an $\mathcal{O}_X(U_{i_0,\dots,i_k})$-module on $M(U_{i_0,\dots,i_k})$ followed by $\mathcal{O}_X(X) \longrightarrow \mathcal{O}_X(U_{i_0,\dots,i_k})$.
Taking the product over all $(i_0,\dots,i_k) \in I^{k+1}$ gives the structure of a cochain complex of $\mathcal{O}_X(X)$-modules, used in the formulation of the lemma.

Proof: As

$$\check{C}^k(\mathcal{U}, M) = \prod_{i \in I^{k+1}} M(U_i)$$

is a finite product and $(\cdot)_f$ commutes with finite products, it is sufficient to show

$$M(U_{i_0 \dots i_k})_f \xrightarrow{\cong} M(U_{i_0 \dots i_k} \setminus V(f))$$

which holds if the $U_{i_0 \dots i_k}$ are q.s. and q.c. They inherit q.s-ness from X and are q.c. being the intersection of q.c. open sets in a q.s. space X.

Proposition 1.2.3: Let X be a q.c. scheme, M a q.c. $\mathcal{O}_X$-module and $f \in \mathcal{O}_X(X)$. Then we have a canonical isomorphism

$$H^*(X, M)_f \xrightarrow{\cong} H^*(X \setminus V(f), M|_{X \setminus V(f)})$$

Proof: Follows from lemma 1.2.3.

Remark: In general there is the machinary of derived functors (from Grothendieck's Tohoku paper) applicable to left exact functors on an Abelian category with sufficiently many injective objects, like R-mod ~~and $Qc(X)$~~ (R any sheaf of rings over any topological space X.) and $Qc(X)$ (X prescheme)

When M is an R-module ~~of~~ on X, the derived functors taken in the categories of R-modules or sheaves of Abelian groups are canonically isomorphic.

When X is a q.c. (or paracompact) scheme these are isomorphic to the cohomology introduced here and also to the derived functor of global sections taken for $Qc(X)$.

In general these are not the same.

1.3 The affinity criterion of Serre

Proposition 1.3.1 (Serre's affinity criterion) For a q.c. scheme X, the following are equivalent:

a) X is affine

b) $H^p(X, M) = 0$ for $p > 0$, $M \in Qc(X)$

c) $H^1(X, \mathcal{J}) = 0$, when $\mathcal{J}$ is a q.c. sheaf of ideals on X.

Remark: In EGA III, this is shown when X is a q.c. and q.s. prescheme with cohomology defined as a derived functor on $\mathcal{O}_X$-mod.

We need the following:

<u>Proposition 1.3.2</u> a) If Z is a q.c. closed subset of a prescheme X, it contains a closed point (proved by induction of affine open subsets needed to cover Z.)

b) If $Z \subseteq X$ is a closed subset of a prescheme X

$$\mathcal{J}(U) := \{ f \in \mathcal{O}_X(U) \mid Z \cap U \subseteq V(f) \}$$

is a q.c. sheaf of ideals.

c) If $\mathcal{N}_1, \mathcal{N}_2 \subseteq \mathcal{M}$ are q.c. subsheaves of the q.c. sheaf of modules $\mathcal{M}$, then

$$(\mathcal{N}_1 \cap \mathcal{N}_2)(U) := \mathcal{N}_1(U) \cap \mathcal{N}_2(U)$$

defines a q.c. sheaf of modules.

<u>Proof</u> b) Let $U \subseteq X$ be q.c and q.s. and $f \in \mathcal{O}_X(U)$. Let $\varphi \in \mathcal{J}(U)$ be s.t. $\varphi|_{U \setminus V(f)} = 0$. Then by q.c.-ness of $\mathcal{O}_X$ there is $n \in \mathbb{N}$ s.t. $f^n \varphi = 0$ (Remark 1.1.2). If $\psi \in \mathcal{J}(U \setminus V(f))$, then by q.c.-ness of $\mathcal{O}_X$ there is $n \in \mathbb{N}$ s.t. $f^n \psi$ extends to some $g \in \mathcal{O}_X(U)$ and fg extends $f^{n+1}\psi$ and is in $\mathcal{J}(U)$, as $(Z \cap U) \setminus V(f) \subseteq V(\psi) \subseteq V(f^{n+1}\psi) = V(g|_{U \setminus V(f)})$ and

$$V(f) \subseteq V(fg)$$

Thus

$$\mathcal{J}(U)_f \xrightarrow{\simeq} \mathcal{J}(U \setminus V(f)).$$

c) Direct verification of $(\mathcal{N}_1 \cap \mathcal{N}_2)(U)_f \cong (\mathcal{N}_1 \cap \mathcal{N}_2)(U \setminus V(f))$ similar as above or by

$$\mathcal{N}_1 \cap \mathcal{N}_2 = \ker(\mathcal{N}_1 \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}/\mathcal{N}_2)$$

<u>Proof of proposition 1.3.1</u>

a) $\Rightarrow$ b) By theorem 1.

b) $\Rightarrow$ c) Trivial

c) $\Rightarrow$ a) For this we first show:

If $\mathcal{M} \subseteq \mathcal{O}_X^n$ is a q.c. submodule, then $H^1(X, \mathcal{M}) = 0$

<u>Proof:</u> Do induction on n. The case $n=0$ is trivial. Let $(\mathcal{O}_X^{n-1})(U)$ be identified with $\{(f_1, \dots, f_{n-1}, 0) \in \mathcal{O}_X(U)^n\} \subseteq \mathcal{O}_X(U)^n$.

By proposition 1.3.2 c), $\mathcal{M}' := \mathcal{M} \cap \mathcal{O}_X^{n-1}$ is q.c. Let $\mathcal{M}'' := \mathcal{M}/\mathcal{M}'$. Then $\mathcal{M}'' \longrightarrow \mathcal{O}_X^1 : (f_1, \dots, f_n) \longrightarrow f_n$ is a monomorphism.

Hence $H^1(X, \mathcal{M}') = 0$ by induction and $H^1(X, \mathcal{M}'') = 0$ by ~~the~~ induction, so by the piece

$$H^1(X, \mathcal{M}') \longrightarrow H^1(X, \mathcal{M}) \longrightarrow H^1(X, \mathcal{M}'')$$

also $H^1(X, \mathcal{M}) = 0$. $\square$

c) $\Rightarrow$ a) Let $R = \mathcal{O}_X(X)$. The adjunction

$$\mathrm{Hom}(X, \mathrm{Spec}\, R) \xrightarrow{\simeq} \mathrm{Hom}(R, \mathcal{O}_X(X)) = \mathrm{Hom}(R, R)$$

then gives a morphism $p : X \longrightarrow \mathrm{Spec}\, R$ corresponding to id_R.

We have to show that p is an isomorphism. For this it is sufficient to show

α) If $f \in R$ is s.t. $X \setminus V(f)$ is affine, then $X \setminus V(f) = p^{-1}(\mathrm{Spec}\, R \setminus V(f))$ is mapped isomorphically to $\mathrm{Spec}\, R \setminus V(f)$ by p

β) The open subsets $X \setminus V(f)$ as in α) cover X

γ) If $(f_i)_{i=1}^n \subset R^n$ are as in α) and $X = \bigcup_{i=1}^n (X \setminus V(f_i))$, then

$$\mathrm{Spec}\, R = \bigcup_{i=1}^n \mathrm{Spec}\, R \setminus V(f_i)$$

Then by β) and q.c.-ness of X it is possible to find (f_i) to which γ) may be applied. Then the open subsets $U_i = \mathrm{Spec}\, R \setminus V(f_i)$ cover $\mathrm{Spec}\, R$ and $p^{-1}(U_i) \xrightarrow[\simeq]{p} U_i$

<u>Proof of α)</u>

If $X \setminus V(f)$ is affine, we have

$$\begin{array}{ccccc} X \setminus V(f) & \xrightarrow{\simeq} & \mathrm{Spec}(\mathcal{O}_X(X \setminus V(f))) & \simeq & \mathrm{Spec}(R_f) \\ \downarrow & & \downarrow & & \wr\wr \\ & & & & \mathrm{Spec}\, R \setminus V(f) \\ X & \xrightarrow{p} & \mathrm{Spec}\, R & & \end{array}$$

proving α).

<u>Proof of β)</u>

Let $U \subseteq X$ be the union of all affine open subsets of the form $X \setminus V(f)$, with f as in α).

Let $Z = X \setminus U$. If $Z \neq \emptyset$, there is a closed point $z \in Z$. We have an affine open neighborhood V of z. Let $Y_1 \subseteq X$ be the closed subset $X \setminus V$ and $Y_2 = \{z\}$ and $Y = Y_1 \cup Y_2$.

The sheaf of ideals $\mathcal{J}$ of functions $f \in \mathcal{O}_X(W)$ s.t. $Y \cap W \subseteq V(f)$ is q.c. by proposition 1.32 b)

By c), we have $H^1(X, \mathcal{J}) = 0$. Let Y also denote the closed subprescheme defined by $\mathcal{J}$ and

$$Y \xrightarrow{i} X$$

its immersion. Then

$$\mathcal{J} = \ker(\mathcal{O}_X \xrightarrow{i^*} i_* \mathcal{O}_Y)$$

We have $\varphi \in \mathcal{O}_Y(Y) = (i_*\mathcal{O}_Y)(X)$ s.t. $\varphi|_{Y_1} = 0$ and $\varphi|_{Y_2} = 1$ as Y is the disjoint union of the closed sets Y_1, Y_2.

By the cohomology sequence

$$\mathcal{O}_X(X) \longrightarrow (i_*\mathcal{O}_Y)(X) \longrightarrow H^1(X, \mathcal{J}) = 0$$

there is $f \in \mathcal{O}_X(X)$ s.t. $f|_Y = \varphi$. Then $Y_1 \subseteq V(f)$ and $z \notin V(f)$. So

$$z \in X \setminus V(f) = (X \setminus Y_1) \setminus V(f) = V \setminus V(f)$$

which gives an affine (V being affine) neighborhood of z of the right form, so $z \in \mathcal{U}$, a contradiction.

Proof of δ: Let $f_1 \ldots f_n$ be as requested. Then

$$\mathcal{O}_X^n \xrightarrow{(f_1 \ldots f_n)} \mathcal{O}_X$$

is epi as the $V_i = X \setminus V(f_i)$ cover X and $\mathcal{O}_X \xrightarrow{f_i} \mathcal{O}_X$ is an isomorphism ~~for~~ on V_i.

Let $\mathcal{M}$ be its kernel. We have

$$\begin{array}{ccccccc} & & R^n & \xrightarrow{(g_i)_{i=1}^n \mapsto \sum g_i f_i} & R & & \\ & & \wr\wr & & & & \\ \mathcal{M}(X) & \longrightarrow & \mathcal{O}_X^n(X) & \xrightarrow{(f_1 \ldots f_n)} & \mathcal{O}_X(X) & \longrightarrow & H^1(\mathcal{M}) \end{array}$$

By what was showed before $H^1(\mathcal{M}) = 0$. So the f_i generate R as an ideal in R and $\bigcap_{i=1}^n V(f_i) = \emptyset$ in Spec R, as desired. □

Remark: The vanishing of $H^1(X, \mathcal{O}_X)$ is not enough for the affinity of X (even when X is a q.c. scheme) as this holds for $X = \mathbb{P}^n_R$.

1.4 Cohomological dimension

Proposition 1.4.1 (Grothendieck) Let X be a scheme (with Noetherian underlying topological space?) ~~Then~~ and $Z \subseteq X$ a closed subset which is a Noetherian topological space. Then

$$H^p(X, \mathcal{M}) = 0$$

when $\mathcal{M} \in \mathcal{Q}c(X)$ s.t. $\mathcal{M}|_{X \setminus Z} = 0$ and $p > \dim(Z)$ (the Krull dimension of Z, which may be ∞.)

Proof: We do induction on $\dim(Z)$. When $\dim(Z) = 0$, then the composition into irreducible components $Z = \bigcup_{i=1}^n Z_i$ looks like $Z_i = \{z_i\}$ where $z_i \in X$ is a closed point. Since the affine open subsets form a topology base on X, there are affine open neighborhoods W_i of z_i s.t. $Z \cap W_i = \{z_i\}$.

Let $M_i := (j_i)_*(M|W_i)$ where $j_i : W_i \hookrightarrow X$ is the open immersion. Let $\hat{M} = \bigoplus_{i=1}^{n} M_i$. We have seen in the corollaries to Theorem 1 that

$$H^p(X, \hat{M}) = 0$$

when $p > 0$ and that $\ker(M \longrightarrow \hat{M})$ vanishes on $W := \bigcup_{i=1}^{n} W_i$. [As M and $\hat{M}$ vanish on $U = X \setminus Z$ and $X = U \cup W$ it follows that $M \longrightarrow \hat{M}$ is mono.]
But $(M_i)_{z_j} = 0$ when $i \neq j$, while $M \longrightarrow M_i$ is an isomorphism on W_i, hence on the stalks at z_i. Hence $M \xrightarrow{\cong} \hat{M}$ and the assertion follows.

<u>Induction step</u> Let $\dim(Z)$ be finite and the assertion being proved for lower dimension. Let $Z = \bigcup_{i=1}^{n} Z_i$ be the decomposition into irreducible components and let $\eta_i \in Z_i$ be the generic point of Z_i. We have $\eta_i \notin Z_j$ by minimality of $Z = \bigcup_{j=1}^{n} Z_j$.
As the affine open subsets ~~ex~~ form a topology base, there are affine open neighborhoods W_i of η_i in X s.t. $W_i \cap Z_j = \emptyset$ when $i \neq j$.
Let $M_i = (j_i)_*(M|W_i)$, where $j_i : W_i \longrightarrow X$ is the open immersion.

<u>Claim</u>

The kernel and cokernel of

$$M \longrightarrow \hat{M} := \bigoplus_{i=1}^{n} M_i$$

vanish outside $\hat{X} = X \setminus (\bigcup_{i=1}^{n} W_i)$
which is equivalent to:

- $M \longrightarrow \hat{M}$ mono and epi outside $\hat{X}$
- $M|_{X \setminus \hat{X}} \longrightarrow \hat{M}|_{X \setminus \hat{X}}$ mono and epi
- $M|_x \longrightarrow \hat{M}|_x$ mono and epi for all $x \in X \setminus \hat{X}$.

As was seen in a corollary after theorem 1, $H^p(X, \hat{M}) = 0$ when $p > 0$, and also $M \longrightarrow \hat{M}$ is a monomorphism outside $\hat{X} = X \setminus (\bigcup_{i=1}^{n} W_i)$. Let $\hat{Z} = Z \cap \hat{X}$, then we have $M|_{X \setminus Z} = \hat{M}|_{X \setminus Z}$, so $M \longrightarrow M'$ is also mono outside Z, hence mono outside $\hat{Z}$.
We claim that (*) $M \longrightarrow M'$ is also epi outside $\hat{Z}$.
When this is assumed for the moment, we have

$$(+) \quad 0 \longrightarrow K \longrightarrow M \longrightarrow \hat{M}^{\varepsilon} \longrightarrow C \longrightarrow 0$$

where $\mathcal{K}$ and $\mathcal{C}$ vanish outside $\hat{\Sigma}$. Since W_i intersects Z_i, $Z_i \not\subseteq \hat{X}$, hence no irreducible component of Z is contained in $\hat{\Sigma}$; hence $\dim(\hat{\Sigma}) < \dim(Z) =: d$. If $d = 0$, this implies $\hat{\Sigma} = \emptyset$, hence $\mathcal{M} \cong \hat{\mathcal{M}}$, has vanishing cohomology in positive degrees. Otherwise the induction assumption implies

$$0 = H^p(X, \mathcal{K}) = H^p(X, \mathcal{C})$$

when $p \geq d$. We split $(+)$ into

$$0 \longrightarrow \mathcal{B} \longrightarrow \hat{\mathcal{M}} \longrightarrow \mathcal{C} \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{M} \longrightarrow \mathcal{B} \longrightarrow 0$$

where $\mathcal{B} = \ker(\hat{\mathcal{M}} \to \mathcal{C}) = \operatorname{im}(\mathcal{M} \longrightarrow \hat{\mathcal{M}})$.

This gives us

$$H^{p-1}(X, \mathcal{C}) \longrightarrow H^p(X, \mathcal{B}) \longrightarrow H^p(X, \hat{\mathcal{M}}),$$

implying that $H^p(X, \mathcal{B}) = 0$ when $p > d$ and

$$H^p(X, \mathcal{K}) \longrightarrow H^p(\mathcal{M}) \longrightarrow H^p(X, \mathcal{B})$$

implying $H^p(X, \mathcal{M}) = 0$ when $p > d$. The vanishing assertion then follows.

It remains to prove our claim that $\mathcal{C}|_{X \setminus \hat{\Sigma}} = 0$. For this, we use that

$$(\%) \qquad (\hat{\mathcal{M}}_i)_x \cong \begin{cases} \mathcal{M}_x & x \in Z \text{ in } W_i \\ 0 & x \notin Z_i \end{cases}$$

When $x \notin Z$, it implies $\hat{\mathcal{M}}_x = 0$, hence $\mathcal{M}_x \longrightarrow \hat{\mathcal{M}}_x$ is surjective. When $x \notin \hat{\Sigma}$, then $x \in Z \cap W_i$ for some i, hence $x \in Z_i$ (as $W_i \cap Z_j = \emptyset$ for $i \neq j$), hence

$$(\mathcal{M}_j)_x \cong \begin{cases} \mathcal{M}_x & j = i \\ 0 & \text{otherwise} \end{cases}$$

and $\mathcal{M}_x \longrightarrow \hat{\mathcal{M}}_x$ is again surjective.

For $(\%)$, note that, in the first case, $\mathcal{M}_i(U) \cong \mathcal{M}(U)$ when $U \subseteq W_i$ and for the second case that $\mathcal{M}_i(U) = 0$ when $U \subseteq X \setminus Z_i$ (as $X \setminus Z \subseteq U \cap W_i \cap (X \setminus Z) = U \cap W_i \cap (X \setminus Z_i)$).

In both cases, the open subsets U with the specified property are a fundamental system of open neighborhoods of x, finishing $(\%)$ and thus finishing the proof.

1.5 Cohomology of morphisms

Cohomology of morphisms (i.e. higher direct images) and functorial properties (w.r.t. the scheme) of cohomology.

Let $f : X \longrightarrow Y$ be a q.c. and separated morphism of preschemes. The set B of q.c. open subsets $U \subseteq Y$ which are schemes forms a topology base on Y (as it contains all affine open subsets.)

By well-known properties of q.c. and of separated morphisms, $f^{-1}(U)$ is a q.c. scheme when $U \in B$. (Note that $f^{-1}(U) \xrightarrow{f} U$ is separated and $U \longrightarrow \operatorname{Spec} \mathbb{Z}$ is, so the composition $f^{-1}(U) \to \operatorname{Spec}\mathbb{Z}$ is.)

We thus have, for any q.c. $\mathcal{O}_X$-module M, a presheaf

(%) $\qquad U \longmapsto H^p(f^{-1}(U), M)$

on B. Sheafifying this presheaf gives a $\mathcal{O}_Y$-module, called the <u>p-th direct image</u> of M under f, and is denoted by $R^p f_* M$. We have a canonical morphism

$$\boxed{(1) \qquad H^p(f^{-1}U, M) \longrightarrow (R^p f_* M)(U) = H^0(U, R^p f_* M)}$$

as a special case of the morphism from a presheaf to its sheafification.

The presheaf structure on (%) is defined as follows: let $V \subseteq U$ be elements of B, and $\mathcal{U}, \mathcal{V}$ the coverings of $f^{-1}(U)$ and $f^{-1}(V)$ by their affine open subsets. Then the morphism of restriction to the intersection with $f^{-1}(V)$ gives

$$\check{C}^*(\mathcal{U}, M) \longrightarrow \check{C}^*(\mathcal{U} \cap f^{-1}V, M)$$

which may be followed by any morphism

$$\check{C}^*(\mathcal{U} \cap f^{-1}(V), M) \longrightarrow \check{C}^*(\mathcal{V}, M)$$

defining

$$H^p(f^{-1}U, M) \cong \check{H}^p(\mathcal{U}, M) \longrightarrow \check{H}(\mathcal{V}, M) \longrightarrow H^p(f^{-1}(V), M),$$

defining the desired presheaf structure.

<u>Proposition 1.5.1</u> a) $R^0 f_* M = f_* M$ (canonically)

b) The $\mathcal{O}_X$-modules $R^p f_* M$ are quasi-coherent

c) ~~[illegible]~~ For a s.e.s. $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, there is a l.e.s. cohom sequence (A, B, C q.c. $\mathcal{O}_X$-modules)

$$
\begin{tikzcd}
0 \arrow[r] & f_*A \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & f_*B \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & f_*C \arrow[d, phantom, "\cong" rotate=90] \\
0 \arrow[r] & R^0f_*A \arrow[r] & R^0f_*B \arrow[r] & R^0f_*C \arrow[dll] \\
 & R^1f_*A \arrow[r] & R^1f_*B \arrow[r] & R^1f_*C \arrow[dll] \\
 & R^2f_*A \arrow[r] & \dots &
\end{tikzcd}
$$

d) When $U \subseteq Y$ is affine (1) becomes an isomorphism.

Proof: Let Φ^p denote the presheaf (%) on B. Then R^pf_*M is $\mathrm{Sheaf}(\Phi^p)$.

a) By the definitions

$$\Phi^0(U) = H^0(f^{-1}U, M) \cong M(f^{-1}(U)) = f_*M(U).$$

thus $\Phi^0 \cong f_*M|_B$.

But $F \xrightarrow{\cong} \mathrm{Sheaf}(F|_B)$ for any sheaf on X, providing the desired isomorphism.

c) By theorem 1, we have a similar long exact sequence of presheaves on B. Sheafifying it gives the desired sequence, as sheafification is an exact functor.

It is easy to see that our construction of R^pf_*M is base-local, i.e. that

$$(R^pf_*M)_U \cong (R^pf|_{f^{-1}U})_*(M|_{f^{-1}U})$$

It is thus possible to assume Y to be affine (q.c.-ness being a local condition, for b)) and $Y = U$ in d).

Recall that $R^pf_*M = \mathrm{Sheaf}(U \longmapsto H^p(f^{-1}(U), M))$, where U ranges over $\mathcal{B} = \{U \subseteq Y \mid U \text{ open and q.c. scheme}\}$. The subset $\hat{\mathcal{B}} = \{Y \setminus V(\varphi) \mid \varphi \in R\} \subseteq \mathcal{B}$ is also a topology base of $Y = \operatorname{Spec} R$, and

$$\mathrm{Sheaf}(\mathcal{G}|_{\hat{\mathcal{B}}}) \cong \mathrm{Sheaf}(\mathcal{G})$$

for any presheaf $\mathcal{G}$ on $\mathcal{B}$. But for $U = Y \setminus V(\varphi) \in \hat{\mathcal{B}}$, we have

$$
\begin{aligned}
H^p(f^{-1}U, M) &= H^p(f^{-1}(Y \setminus V(\varphi)), M) \\
&= H^p(X \setminus V(f^*\varphi), M) \\
&\cong H^p(X, M)_\varphi \\
&= M_\varphi
\end{aligned}
$$

with $M = H^p(X, M)$, by proposition 1.2.3.

It follows that $R^p f_* M$ is canonically isomorphic to the sheafification of

$$\mathcal{U} = Y \setminus V(\varphi) \longmapsto M_\varphi$$

which we call $\hat{M}$. By definition of $\hat{M}$, it is a q.c. $\mathcal{O}_Y$-module.
Then (b) follows as $M \cong \hat{M}(Y)$.
(d) follows as well □

In general the morphism (1)

$$(1) \qquad H^p(X, M) \longrightarrow (R^p f_* M)(Y)$$

fails to be an isomorphism
(Even the special case of (1) of "restriction to the fibers" is not in general an isomorphism. $f^{-1}(\{y\}) := X \times_Y \mathrm{Spec}(k(y))$)

There is another morphism

$$\boxed{H^p(Y, f_* M) \longrightarrow H^p(X, M).}$$

when Y is a scheme, called the "pull-back". It is also not in general an isomorphism unless f is affine.
It can be defined as follows: let $\mathcal{U}_Y$ be the covering of Y by its affine subsets, and $\mathcal{V}$ an affine refinement of $f^{-1}\mathcal{U}_Y$. Let $\mathcal{U}_X$ be the covering of X by its affine subsets.
Then

$$H^p(Y, f_* M) = \check{H}^p(\mathcal{U}_Y, f_* M) \overset{\text{lem 1.2.2}}{=} \check{H}^p(f^{-1}\mathcal{U}_Y, M)$$

$$\xrightarrow{\text{refinement}} \check{H}^p(\mathcal{V}, M) \underset{\text{refinement}}{\overset{\cong}{\longleftarrow}} \check{H}^p(\mathcal{U}_X, M) = H^p(X, M)$$

So we have two morphisms

$$(+) \qquad H^p(X, M) \longrightarrow (R^p f_* M)(Y) = H^0(Y, R^p f_* M)$$

$$(*) \qquad H^p(Y, f_* M) \longrightarrow H^p(X, M)$$

There is also an inverse image $\mathcal{O}_Y\text{-mod} \xrightarrow{f^*} \mathcal{O}_X\text{-mod}$, preserving quasi-coherence, which is left adjoint to f_*, thus $N \longmapsto f_* f^* N$ gives

$$H^p(Y, N) \longrightarrow H^p(Y, f_* f^* N) \xrightarrow{(*)} H^p(X, f^* N)$$

In general, (+) and (*) will fail to be isomorphisms but are part of the ~~Leray sequence~~ Leray Spectral sequence:

$$E_2^{p,q} = H^p(Y, R^q f_* M)$$

converging to $H^{p+q}(X, M)$, with filtration
Note: $E_2^{p,0} = H^p(Y, f_* M)$

$$E_2^{0,q} = H^0(Y, R^p f_* M) = (R^p f_* M)(Y)$$

This means the following: one has a sequence $(E_r^{p,q})_{r=2}^{\infty}$ of doubly graded abelian groups (or R-modules) equipped with morphisms, called the <u>differentials of the spectral sequence</u>

$$d_k : E_k^{p,q} \longrightarrow E_k^{p+k,\, q+1-k}$$

of total degree 1, s.t. $d_k^2 = 0$, and s.t.

$$E_{k+1} = H(E_k, d_k)$$

i.e.

$$(\eta) \qquad E_{k+1}^{p,q} = \ker(d_k : E_k^{p,q} \longrightarrow E_k^{p+k,\, q+1-k}) \Big/ \operatorname{im}(d_k : E_k^{p-k,\, q-1+k} \longrightarrow E_k^{p,q})$$

Note that in particular $E_k^{p,q} = 0$ when $E_2^{p,q} = 0$. So when $E_2^{p,q}$ is supported in the first quadrant, the same holds for all $E_k^{p,q}$, so for $k > q+1$

$$(\eta') \qquad E_{k+1}^{p,q} = \operatorname{Coker}(E_k^{p-k,\, q+k-1} \xrightarrow{d_k} E_k^{p,q})$$

and for $k > p$

$$(\eta'') \qquad E_{k+1}^{p,q} = \ker(E_k^{p,q} \xrightarrow{d_k} E_k^{p+k,\, q+1-k})$$

In particular, for $k > p$, $k > q+1$

$$E_{k+1}^{p,q} = E_k^{p,q}.$$

So for first quadrant spectral sequences, we can define the ∞-page as

$$E_\infty^{p,q} = \lim_k E_k^{p,q}$$

We say that the spectral sequence converges to its limit L^* (for Leray, $L^* = H^*(X, M)$) if there is a filtration $F^p L^*$ on L^* (i.e. $L^* = F^0 L^* \supseteq F^1 L^* \supseteq \dots$) with

$$E_\infty^{p,q} \cong F^p L^{p+q} \big/ F^{p+1} L^{p+q}$$

We thus get

$$F^1 L^p = \ker(L^p \longrightarrow E_\infty^{0,p}) = \ker(L^p \longrightarrow E_2^{0,p})$$

(as $E_\infty^{0,p} \subseteq E_2^{0,p}$ by (η''))

This has (+) as a special case.

Also, using $F^{p+1} L^p = 0$,

$$E_2^{p,0} \longrightarrow\!\!\!\!\to E_\infty^{p,0} \simeq F^p L^p.$$

The resulting morphism $E_2^{p,0}$ has $(*)$ as a special case.

(There is also a Leray spectral sequence $R^p g_* R^q f_* M \longrightarrow R^{p+q}(gf)_* M$)

1.6 Affine morphisms

<u>Proposition 1.6.1</u> Let X be a q.s. prescheme and Y an arbitrary prescheme. For a morphism $f: X \longrightarrow Y$ (that is q.c. and q.s), the following conditions are equivalent:

a) If $U \subseteq Y$ is open affine, $f^{-1}(U)$ is affine

b) It is possible to cover Y by affine open U s.t. $f^{-1}(U)$ affine

c) f is separated and q.c. and we have $R^p f_* \mathcal{M} = 0$ for all q.c. $\mathcal{O}_X$-modules ~~M~~ and $p > 0$.

<u>Proof</u>: a) $\Rightarrow$ b) Trivial

b) $\Rightarrow$ c) Follows easily from Theorem 1 and proposition 1.5.1

c) $\Rightarrow$ a) If $U \subseteq Y$ is open affine, then if $\mathcal{M} \in Qc(f^{-1}U)$ it extends to X (we will show this later), and by proposition 1.5.1

$$H^p(f^{-1}U, \mathcal{M}) \cong (R^p f_* \mathcal{M})(U) = 0$$

By Serre's criterion, $f^{-1}U$ is affine.

It remains to show

<u>Lemma</u>: ~~Let X be a q.s. pret~~

Let X be a q.s. prescheme and $\mathcal{M}$ a q.c. $\mathcal{O}_U$-module, where $U \subseteq X$ is a q.c. open subset. Then there is a q.c. $\mathcal{O}_Y$-module $\tilde{\mathcal{M}}$ s.t. $\tilde{\mathcal{M}}|_U \cong \mathcal{M}$.

<u>Proof</u>: The open immersion $V \xrightarrow{j} X$ being q.s. and q.c. (X is ~~separated hence~~ q.s. and V is q.c.), hence $\tilde{\mathcal{M}} := j_* \mathcal{M}$ is q.c. by what was recalled in 1.1 and $\tilde{\mathcal{M}}|_V \cong \mathcal{M}$.

<u>Definition 1.6.1</u>: α) A morphism $f: X \longrightarrow Y$ between preschemes is called <u>affine</u> if it satisfies the equivalent conditions a) and b).

β) It is called <u>finite</u> if in addition, for any affine open $U \subseteq Y$, $\mathcal{O}_X(f^{-1}U)$ is a f.g. $\mathcal{O}_Y(U)$-module.

Equivalently (lemma 1.1.1) this holds for sufficiently many U.

<u>Remark</u>: In other words

f finite $\Longleftrightarrow$ f affine and $f_*\mathcal{O}_X$ is a locally finitely generated $\mathcal{O}_Y$-module (see 1.1)

$\underset{\text{if } Y \text{ loc. Noeth}}{\Longleftrightarrow}$ f affine and $f_*\mathcal{O}_X$ a coherent $\mathcal{O}_Y$-module.

<u>Corollary 1.6.1</u>: If $f: Y \longrightarrow X$ is an affine morphism and $\mathcal{M}$ a q.c. $\mathcal{O}_Y$-module then $R^p f_* \mathcal{M} = 0$ when $p > 0$ and

$$(1.5.3) \qquad H^p(X, f_* \mathcal{M}) \longrightarrow H^p(Y, \mathcal{M})$$

is an isomorphism.

<u>Proof</u>: The first assertion follows from the fact that $R^p f_* \mathcal{M}$ is the sheafification of $\mathcal{U} \longmapsto H^p(f^{-1}\mathcal{U}, \mathcal{M})$, which when $p > 0$ vanishes on the topology base of affine open $\mathcal{U}$ by Theorem 1.

When X is a q.c. scheme, then $H^p(X, f_*\mathcal{M}) \cong \check{H}^p(\mathfrak{U}, f_*\mathcal{M}) \cong \check{H}^p(f^{-1}\mathfrak{U}, \mathcal{M}) \cong H^p(Y, \mathcal{M})$, which was our construction of the morphism in 1.5.3. The last map is an iso since affiness of f implies that $f^{-1}\mathfrak{U}$ is an affine open cover.

This proves the second ~~assertion~~ assertion under the assumptions under which we constructed the relevant cohomology groups. Using the general derived functor construction, it follows from the Leray SS

$$E_2^{pq} = H^p(X, R^q f_* \mathcal{M}) \longrightarrow H^{p+q}(Y, \mathcal{M})$$

and the vanishing of $R^q f_* \mathcal{M}$ when $q > 0$, $\mathcal{M}$ is q.c. and f affine. $\square$

When $\mathcal{A}$ is a q.c. $\mathcal{O}_Y$-algebra, one has schemes

$$\operatorname{Spec} \mathcal{A}(\mathcal{U}) \longrightarrow \operatorname{Spec}(\mathcal{O}_Y(\mathcal{U})) \cong \mathcal{U}$$

when $\mathcal{U}$ is an affine open subset of Y, and when $V \subseteq \mathcal{U}$ then $\operatorname{Spec} \mathcal{A}(V)$ is isomorphic to the preimage of V under $\operatorname{Spec} \mathcal{A}(\mathcal{U}) \longrightarrow \mathcal{U}$. This allows one to glue the different $\operatorname{Spec} \mathcal{A}(\mathcal{U})$ for $\mathcal{U} \subseteq Y$ open affine to one prescheme

<u>$\operatorname{Spec} \mathcal{A}$</u> $\longrightarrow Y$. &

This also admits the following description:

<u>points</u>: pairs $(y, \mathfrak{p})$ where $y \in Y$, $\mathfrak{p} \in \operatorname{Spec} \mathcal{A}_y$ s.t. $\mathfrak{p} \cap \mathcal{O}_{Y,y} = \mathfrak{m}_{Y,y}$

<u>topology base</u> For $\mathcal{U}$ affine, $\lambda \in \mathcal{A}(\mathcal{U})$,
$\Omega(\mathcal{U}, \lambda) = \{(y, \mathfrak{p}) \mid y \in \mathcal{U}$ and $\mathfrak{p}$ is s.t. (image of λ in $\mathcal{A}_y$) $\notin \mathfrak{p}\}$

local rings: $\mathcal{O}_{\underline{\mathrm{Spec}}\,A,\,[y,\mathfrak{p}]} = (A_y)_{\mathfrak{p}}$

sections: $\mathcal{O}_{\underline{\mathrm{Spec}}\,A}(U) = \{ (f_{[y,\mathfrak{p}]}) \in \prod_{[y,\mathfrak{p}]\in U} \mathcal{O}_{\underline{\mathrm{Spec}}\,A,\,[y,\mathfrak{p}]} \mid \text{c.c.} \}$

where the coherence condition (c.c.) is:

U may be covered by $\Omega(V,\lambda)$ with $V \subseteq Y$ affine, $\lambda \in A(V)$, on which there is $\varphi \in A(V)_\lambda$ s.t. for all $(y,\mathfrak{p}) \in \Omega(V,\lambda)$, $f_{[y,\mathfrak{p}]}$ is the image of φ under

$$A(V)_\lambda \longrightarrow (A_y)_\lambda \longrightarrow (A_y)_{\mathfrak{p}} = \mathcal{O}_{\underline{\mathrm{Spec}}\,A,\,[y,\mathfrak{p}]}$$

One has a bijection

$$(2)\quad \mathrm{Hom}_{Y\text{-PreSch}}(X, \underline{\mathrm{Spec}}\,A) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{O}_Y\text{-alg.}}(A, \xi_* \mathcal{O}_X)$$

when $\xi : X \longrightarrow Y$ is any morphism of preschemes, by gluing the sets

$$\mathrm{Hom}(\xi^{-1}U, \mathrm{Spec}(A(U))) \xrightarrow{\simeq} \mathrm{Hom}(A(U), \mathcal{O}_X(\xi^{-1}(U)))$$

Proposition 1.6.2 a) The morphism $\underline{\mathrm{Spec}}\,A \longrightarrow Y$ is affine

b) (2) is a bijection

c) $\xi : X \longrightarrow Y$ is affine if and only if $A := \xi_* \mathcal{O}_X$ is a q.c. $\mathcal{O}_Y$-module and the morphism $X \longrightarrow \underline{\mathrm{Spec}}\,\xi_* \mathcal{O}_Y$ corresponding to $\mathrm{id}_A \in \mathrm{Hom}(A, \xi_* \mathcal{O}_X)$ under (2) is an isomorphism.

Proof: Mostly an easy consequence of analogous results in absolute case.

Corollary 1.6.1 (to (b)): For $f : \tilde{Y} \longrightarrow Y$ any morphism,

$$\underline{\mathrm{Spec}}(f^* A) \simeq \underline{\mathrm{Spec}}\,A \times_Y \tilde{Y}$$

Here $f^* : \mathcal{O}_Y\text{-algebras} \longrightarrow \mathcal{O}_{\tilde{Y}}\text{-algebras}$ is left adjoint to $f_* : \mathcal{O}_{\tilde{Y}}\text{-algebras} \longrightarrow \mathcal{O}_Y\text{-algebras}$.

1.7 The relation between H^1 and toseurs

If the group G acts on the set X in a simply transitive way, then any $x \in X$ defines a bijection

$$G \longrightarrow X$$
$$g \longmapsto gx$$

which is an iso in the category of sets with a G-action.

Definition 1.7.1: Let X be any topological space and $\mathcal{G}$ a sheaf of groups on X. A $\mathcal{G}$-torseur or $\mathcal{G}$-principal homogeneous space is a sheaf $\mathcal{X}$ of ~~s~~ sets on X with a morphism

$$\mathcal{G} \times \mathcal{X} \longrightarrow \mathcal{X}$$

of sheaves of sets indicated by $(g, \xi) \mapsto g \cdot \xi$, with the property that $1\!\!1_{\mathcal{G}(u)} \cdot \xi = \xi$ and $(g(h\xi)) = (gh)\xi$ and s.t. $\mathcal{X}_x \neq \emptyset$ for any $x \in X$ and the action of $\mathcal{G}_x$ on $\mathcal{X}_x$ is simply transitive.

The torseur is called trivial if $\mathcal{X}(X) \neq \emptyset$.

Example 1.7.1: $\mathcal{G}$ with its group multiplication is a $\mathcal{G}$-torseur. This is ~~called the~~ trivial as $1 \in \mathcal{G}(X)$.

Remark 1.7.1: a) If $x \in \mathcal{X}(X)$, then

$$\mathcal{G} \longrightarrow \mathcal{X}$$
$$g \in \mathcal{G}(u) \longmapsto g \cdot (x|_u)$$

is an isomorphism as it's an isomorphism on stalks.

Hence the ~~nam~~ terminology "trivial".

b) If $\mathcal{X}$ and $\mathcal{Y}$ are $\mathcal{G}$-tenseurs, then any morphism $\mathcal{X} \xrightarrow{f} \mathcal{Y}$ of sheaves of sets compatible with the $\mathcal{G}$-action induces isomorphisms on stalks

Example 1.7.2: a) If $\mathcal{L}$ is a line bundle (locally free $\mathcal{O}_X$-module of rank 1) then

$$U \longmapsto \mathcal{L}^*(U) = \{\lambda \in \mathcal{L}(U) \mid \mathcal{O}_U \xrightarrow{\lambda} \mathcal{L}|_U \text{ is iso}\}$$
$$= \{\lambda \in \mathcal{L}(U) \mid \text{img. of } \lambda \text{ gen. of } \mathcal{L}|_U\}$$

is an $\mathcal{O}_X^*$-tenseur.

Example 1.7.2: a) For a line bundle $\mathcal{L}$ on a locally ringed space X, and an open $\mathcal{U} \subseteq X$, we put

$$V(\ell) = \{ x \in \mathcal{U} \mid (\text{image of } \ell \text{ in } \mathcal{L}_x) \in m_x \mathcal{L}_x \}$$
$$= V\left(\frac{\ell}{\lambda}\right)$$

for all $\ell \in \mathcal{L}(\mathcal{U})$, where $\lambda \in \mathcal{L}(\mathcal{U})$ can be any free generator of $\mathcal{L}|\mathcal{U}$.

Define

$$\mathcal{L}^*(\mathcal{U}) = \{ \ell \in \mathcal{L}(\mathcal{U}) \mid V(\ell) = \emptyset \}$$
$$= \{ \ell \in \mathcal{L}(\mathcal{U}) \mid \ell \text{ is a free generator of } \mathcal{L}|\mathcal{U} \}$$

Obviously, this is a sheaf ~~of~~ sets, on which the sheaf $\mathcal{O}_X^*$ (defined by $\mathcal{O}_X^*(\mathcal{U}) = \mathcal{O}_X(\mathcal{U})^*$) acts via

$$(f, \ell) \longmapsto f \cdot \ell.$$

Obviously this is a $\mathcal{O}_X^*$-torseur and the line-bundle is trivial iff $\mathcal{L}^*$ has a global section (i.e. is a trivial torseur). It is obvious that for an isomorphism $\mathcal{L} \xrightarrow[\varphi]{\cong} M$ of line bundles we have an isomorphism

$$\mathcal{L}^* \xrightarrow{\Phi} M^*$$

sending $\ell \in \mathcal{L}^*(\mathcal{U})$ to $\varphi(\ell) \in M^*(\mathcal{U})$. Vice versa if $\mathcal{L}^* \xrightarrow{\Phi} M^*$ is an (iso-)morphism of $\mathcal{O}_X^*$-torseurs, there is a unique isomorphism $\mathcal{L} \xrightarrow{\varphi} M$ of line bundles s.t. $\varphi(\ell) = \Phi(\ell)$ for all open $\mathcal{U} \subseteq X$ and $\ell \in \mathcal{L}^*(\mathcal{U})$.

We get a bijection between ~~the~~ isomorphism classes of line bundles and the isomorphism classes of $\mathcal{O}_X^*$-torseurs: If $\mathfrak{X}$ is an $\mathcal{O}_X^*$-torseur, we define $\mathcal{L}$ as follows:

$$\mathcal{L} = \mathrm{Sheaf}\left(\mathcal{U} \longmapsto \{ (f, \xi)/\sim \} \right)$$

where $f \in \mathcal{O}_X(V)$, $\xi \in \mathfrak{X}$ and

$$(f, \xi) \sim (g, v) \quad \text{iff} \quad f = g \cdot \frac{v}{\xi}$$

and addition of $(f, \xi)/\sim$ and $(g, v)/\sim$ is

$$\left(g \cdot \frac{v}{\xi} + f, \xi \right)/\sim$$

and multiplication with $h \in \mathcal{O}_X(V)$ is

$$h(f, \xi)/\sim \; = (fh, \xi)/\sim.$$

This gives the desired line bundle.

b) Let

$$(Gl_n(\mathcal{O}_X))(U) = \{ g \in Mat(n,n; \mathcal{O}_X(U)) \mid V(\det(g)) = \emptyset \}$$
$$= \{ g \in Mat(n,n, \mathcal{O}_X(U)) \mid g \text{ has inverse matrix} \}$$

We have a similar bijection

$$\left(\begin{array}{l} \text{Isomorphism classes of} \\ \text{n-dim vector bundles on } X \end{array} \right) \longleftrightarrow \left(\begin{array}{l} \text{Isomorphism classes of} \\ Gl_n(X)\text{-torseurs} \end{array} \right)$$

$$\mathcal{E}/\sim \longmapsto \{ \vec{e} = (e_i)_{i=1}^n \in \mathcal{E}(U)^n \mid e \text{ is a vector of free generators of } \mathcal{E}|_U \} /\sim$$

where $Gl_n(\mathcal{O}_X)$ acts by right multiplication with the column vector $\vec{e}$.

Let $\mathcal{G}$ be a sheaf of abelian groups, $\mathcal{X}$ a $\mathcal{G}$-torseur.
$\mathcal{U}: X = \bigcup_{i \in I} U_i$ a covering s.t. $\mathcal{X}|_{U_i}$ is trivial.
(We will say that $\mathcal{X}$ is trivial on $\mathcal{U}$.)
If $\xi_i \in \mathcal{X}(U_i)$, then there is a unique $\delta = \xi_i|_{U_{ij}} - \xi_j|_{U_{ij}}$ in $\mathcal{G}(U_{ij})$ s.t. $\delta + \xi_j|_{U_{ij}} = \xi_i|_{U_{ij}}$.
Define $\psi = (\psi_{ij})_{i,j \in I \times I} \in \check{C}^1(\mathcal{U}, \mathcal{G})$ by

$$\psi_{ij} = \xi_i|_{U_{ij}} - \xi_j|_{U_{ij}}$$

This is a "cocycle", since

$$(\check{d}\psi)_{ijk} = \psi_{jk}|_{U_{ijk}} - \psi_{ik}|_{U_{ijk}} + \psi_{ij}|_{U_{ijk}}$$
$$= ((\xi_j - \xi_k) - (\xi_i - \xi_k) + (\xi_i - \xi_j))|_{U_{ijk}}$$
$$= 0$$

(even in non-abelian case)

Proposition 1.7.1: a) The cohomology class $[\psi] \in \check{H}^1(\mathcal{U}, \mathcal{G})$ depends only on the isomorphism class of the $\mathcal{G}$-torseur $\mathcal{X}$, not the choice of trivializations.
b) One gets a bijection between the isomorphism classes of $\mathcal{G}$-torseurs trivial on $\mathcal{U}$ and $\check{H}^1(\mathcal{U}, \mathcal{G})$.

Proof a) If $\hat{\xi}$ is another trivialization yielding

$$\hat{\psi}_{ij} = \hat{\xi}_i|_{U_{ij}} - \hat{\xi}_j|_{U_{ij}},$$

then $\hat{\xi}_i = \gamma_i + \xi_i$, where $\gamma_i = \hat{\xi}_i - \xi_i \in \mathcal{G}(U_i)$ and

$$\hat{\psi} = \check{d}(\gamma) + \psi.$$

b) It is clear that we have a map from isomorphism classes of torseurs trivial on $\mathcal{U}$ to $\check{H}^1(\mathcal{U}, \mathcal{G})$

Let $\eta \in \check{H}^1(\mathcal{U}, \mathcal{G})$ and let $\psi \in \check{Z}^1(\mathcal{U}, \mathcal{G})$ be a representative of η and let

$$\mathcal{X}_\psi(V) = \{\gamma \in \check{C}^2(V \cap \mathcal{U}) \mid d\gamma = \psi|_V\}$$

on which $g \in \mathcal{G}(V)$ acts as

$$(\gamma_i) \in \prod_{i \in I} \mathcal{G}(U_i \cap V) \longmapsto (g + \gamma_i)_{i \in I}$$

If $\hat{\psi} = \psi + dg$ with $g \in \check{Z}^0(\mathcal{U}, \mathcal{G})$, then we have an isomorphism of $\mathcal{G}$-torseurs

$$\mathcal{X}_\psi \longrightarrow \mathcal{X}_{\hat{\psi}}$$

$$\gamma \in \mathcal{X}_\psi(V) \longmapsto \gamma + g|_{U \cap V}.$$

Therefore we get a map in the opposite direction:

$$\check{H}^1(\mathcal{U}, \mathcal{G}) \longrightarrow \{\text{isomorphism classes of } \mathcal{G}\text{-torseurs}\}$$

$$[\psi] \longmapsto \text{isomorphism class of } \mathcal{X}_\psi.$$

The torseur $\mathcal{X}_\psi$ is trivial on U_i as U_i is a member of $\mathcal{U} \cap U_i$.
One easily checks that these two maps are inverse to each other:
if an arbitrary $\mathcal{X}$ is given, and $(\xi_i) \in \prod_{i \in I} \mathcal{X}(U_i)$ and $\psi_{ij} = \xi_i - \xi_j$
then

$$\mathcal{X}(V) \xrightarrow{\simeq} \mathcal{X}_\psi(V)$$

$$v \in \mathcal{X}(V) \longmapsto g \in \mathcal{X}_\psi(V) \subseteq \check{Z}^0(U \cap V_i, \mathcal{G}),\quad g_i = v - \xi_i$$

$$v \longleftarrow g$$

the unique $v \in \mathcal{X}(V)$ s.t. $v|_{U_i \cap V} = g + \xi_i$

<u>Remark</u>: One defines $H^1(X, \mathcal{G})$ as the set of isomorphism classes of $\mathcal{G}$-torseurs on X. This is a pointed set, the base-point being the isomorphism class of the trivial torseur.
Thus, the isomorphism classes of line bundles (resp. n-dim vector bundles) on X are in canonical bijection with $H^1(X, \mathcal{O}_X^*)$ (resp $H^1(X, GL_n(\mathcal{O}_X))$)

<u>Corollary 1.7.1</u>: If X is an affine prescheme, M a q.c. $\mathcal{O}_X$-module and $\mathcal{X}$ an M-torseur on X, then $\mathcal{X}$ is trivial.

<u>Definition 1.7.2</u>: If X is an S-prescheme, we have a homomorphism of sheaves of abelian groups

$$\mathcal{O}_X^* \xrightarrow{d\log} \Omega_{X/S}: \quad d\log f = \frac{d_{X/S} f}{f}$$

defining a morphism of (e.g. Čech) cohomology groups
$$H^1(X, \mathcal{O}_X^*) \longrightarrow H^1(X, \Omega_{X/S})$$
The image of this map of the element of $H^1(X, \Omega_{X/S})$ associated to the torseur $\mathcal{L}^*$ by proposition 1.7.1 is called the <u>first Chern class</u> $c_1(\mathcal{L}) \in H^1(X, \Omega_{X/S})$

<u>Definition 1.7.3</u>: Let $\mathcal{V}, \mathcal{W}$ be vector bundles on the locally ringed space $\mathcal{X}$. An extension of $\mathcal{W}$ by $\mathcal{V}$ is a short exact sequence
$$0 \longrightarrow \mathcal{V} \xrightarrow{a} \mathcal{E} \xrightarrow{b} \mathcal{W} \longrightarrow 0.$$
A morphism of extensions is a commutative diagram
$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{V} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{W} & \longrightarrow & 0 \\ & & \| & & \downarrow \varepsilon & & \| & & \\ 0 & \longrightarrow & \mathcal{V} & \longrightarrow & \tilde{\mathcal{E}} & \longrightarrow & \mathcal{W} & \longrightarrow & 0 \end{array}$$

<u>Proposition</u>: Associating to any extension the tenseur over $\mathcal{H}om(\mathcal{W}, \mathcal{V})$ ($\mathcal{H}om(\mathcal{V}, \mathcal{W})(U) = Hom_{\mathcal{O}_U}(\mathcal{V}|_U, \mathcal{W}|_U)$) given by
$$\mathcal{X}(V) = \{ \xi|_V \xrightarrow{\pi} \mathcal{V}|_W \mid \pi a = id_{\mathcal{V}|_V} \}$$
is an equivalence of categories between $Ext(\mathcal{W}, \mathcal{V})$ and $\{ \mathcal{H}om(\mathcal{W}, \mathcal{V})$-torseurs on $X \}$.

2 Cohomology of projective spaces

2.1 Regular sequences and the Koszul complex

<u>Remark</u>: a) For a cochain complex (C^*, d_{C^*}), we define its <u>shift</u> $C^*[p]$ as the cochain complex
$$(C^*[p])^q := C^{p+q}$$
$$d^q_{C^*[p]} := (-1)^p d_{C^*}^{p+q}$$

Obviously
$$H^q(C^*[p]) = H^{p+q}(C^*)$$
b) Let $\varphi: C^* \xrightarrow{\varphi} \tilde{C}^*$ be a morphism of cochain complexes. We define its <u>cone</u> $Cone(\varphi)$ as the cochain complex

(1) $\mathrm{Cone}(\varphi)^p = \tilde{C}^p \oplus C^{p+1}$

$$d_{\mathrm{Cone}(\varphi)}(\tilde{c}, c) := (d_{\tilde{C}^*}(\tilde{c}) + \varphi(c), -d_{C^*}(c)),$$

which satisfies $d^2=0$ as $d_{\tilde{C}^*}^2 = 0$, $d_{C^*}^2 = 0$, ~~[illegible]~~ and

$\varphi(-d_{C^*}(c)) + d_{\tilde{C}^*}(\varphi(c)) = 0.$

We have a ses of cochain complexes

$$0 \longrightarrow \tilde{C}^* \longrightarrow \mathrm{Cone}(\varphi) \longrightarrow C[1]^* \longrightarrow 0$$
$$\hat{c} \longmapsto (\hat{c}, 0)$$
$$(\hat{c}, c) \longmapsto c$$

The connecting homomorphism

$$H^{p+1}(C^*) = H^p(C[1]^*) \longrightarrow H^{p+1}(\hat{C}^*)$$

coincides with the morphism induced by φ, so

(2) $$\cdots \longrightarrow H^p(C) \xrightarrow{\varphi} H^p(\hat{C}) \longrightarrow H^p(\mathrm{Cone}(\varphi)) \longrightarrow H^{p+1}(C) \longrightarrow \cdots$$

It follows that φ induces isomorphisms in cohomology iff $\mathrm{Cone}(\varphi)$ has vanishing cohomology.

Definition 2.1.1: Let R be a ring and M an R-module. A sequence $(x_0, \dots, x_n)$ of elements of R is called M-regular if for $0 \le i \le n$, the map

$$M/(x_0M + \dots + x_{i-1}M) \xrightarrow{\cdot x_i} M/(x_0M + \dots + x_{i-1}M)$$

is an injective map.

It is a regular sequence if it's M-regular for $M = R$.

Example 2.1.1 a) If S is any ring and $R = S[X_0, \dots, X_n]$, then the sequence $(X_0, \dots, X_n)$ is R-regular. Indeed, injectivity of $R \xrightarrow{\cdot X_0} R$ is obvious and the higher steps are reduced to this one as

$$R/(X_0R + \dots + X_{i-1}R) \cong S[X_i, \dots, X_n].$$

b) The sequence $(1, 0)$ is always regular, but $(0, 1)$ is only regular when $R = 0$.

So we see the order of the x_i matters in definition 2.1.1.

Definition 2.1.2: Let R be a ring, M an R-module and $x = (x_0, \dots, x_n)$ a sequence of elements of R. Let $\underline{n} = \{0, \dots, n\}$ and $C^m(x, M)$ the ~~map f~~ collection of maps $f : \underline{n}^m \longrightarrow M$ satisfying the properties

a) $f(i_1, \dots, i_m) = 0$ when $i_k = i_l$ for some $k \neq l$

b) $f(i_{\pi(1)}, \dots, i_{\pi(m)}) = \operatorname{sgn}(\pi) f(i_1, \dots, i_m)$

The differential $d : C^m(x, M) \longrightarrow C^{m+1}(x, M)$ is given by

$$df = \sum_{j=0}^{m} (-1)^j d_j f$$

where

$$d_j f(i_1, \dots, i_{m+1}) = x_{i_{j+1}} f(i_1, \dots, \hat{i}_{j+1}, \dots, i_{m+1})$$

Remark 2.1.2: It is easy to see that the d_i satisfy relations similar to (1.2.2) and (1.2.4) hence d preserves the assymmetry condition and is a differential.

Remark 2.1.3: a) This is a cochain complex of R-modules, called the Koszul-complex $C^*(x, M)$.

b) A s.e.s. of modules gives a s.e.s. of Koszul complexes.

c) $C^*(x, M)$ vanishes in degrees < 0 and $> n+1$ because of the assymmetry condition.

We have

$$H^0(x, M) = \bigcap_{i=0}^{n} \ker(M \xrightarrow{x_i} M)$$

$$H^{n+1}(x, M) \cong M / (x_0 M + \dots + x_n M)$$

where we defined

$$H^*(x, M) := H^*(C^*(x, M))$$

Example a) For $x = ()$ the empty sequence,

$$C^*((), M) = (\overset{-1}{0} \longrightarrow \overset{0}{M} \longrightarrow \overset{1}{0} \longrightarrow \dots)$$

b) For $x = (x_0)$,

$$C^*((x_0), M) = (\overset{-1}{0} \longrightarrow \overset{0}{M} \xrightarrow{\cdot x_0} \overset{1}{M} \longrightarrow \overset{2}{0} \longrightarrow \dots)$$

c) For $x = (x_0, x_1)$

$$C^*((x_0, x_1), M) = (\overset{-1}{0} \longrightarrow \overset{0}{M} \xrightarrow{(x_0, x_1)} \overset{1}{M \oplus M} \longrightarrow \overset{2}{M} \longrightarrow \overset{3}{0} \longrightarrow)$$

$$m \longmapsto (x_0 m, x_1 m)$$

$$(\mu, \nu) \longmapsto x_0 \nu - x_1 \mu$$

<u>Remark 2.1.3</u> c) We have a canonical isomorphism

$$(3) \quad C^*((x_0 \ldots x_n), M) \xrightarrow{\cong} \mathrm{Cone}\left(\begin{array}{c} C^*((x_0 \ldots x_{n-1}), M) \\ \downarrow \cdot x_n \\ C^*((x_0, \ldots x_{n-1}), M) \end{array} \right)[-1]$$

$$f \longmapsto \left(f(n, -, \ldots, -)|_{(n-1)^{m-1}},\ f|_{(n-1)^m} \right)$$

<u>Fact 2.1.1</u> a) Let $0 \leq i \leq n$. Then $(x_0 \ldots x_n)$ is M-regular iff $(x_0 \ldots x_{i-1})$ is M-regular and $(x_i, \ldots x_n)$ is $M/(x_0M + \ldots + x_{i-1}M)$ - regular.

b) A sequence $(x_0 \ldots x_n)$ is M-regular iff $H^j((x_0 \ldots x_i), M) = 0$ for $0 \leq i \leq n$ and $j \neq i+1$.

<u>Proof</u> a) Obvious

b) The "only if"-part follows by induction on i, using the long exact cohomology sequence (2) resulting from (3):

$$\ldots \to H^{j-1}((x_0 \ldots x_{i-1}), M) \xrightarrow{\cdot x_i} H^{j-1}((x_0 \ldots x_{i-1}), M) \to H^j((x_0 \ldots x_i), M)$$
$$\to H^j((x_0 \ldots x_{i-1}), M) \xrightarrow{\cdot x_i} H^j((x_0 \ldots x_{i-1}), M) \to \ldots$$

By the induction assumption makes this reduce to

$$0 \to H^i((x_0 \ldots x_i), M) \to M/(x_0M + \ldots + x_{i-1}M) \xrightarrow{\cdot x_i} M/(x_0M + \ldots + x_{i-1}M)$$
$$\to H^{i+1}((x_0 \ldots x_i), M) \to 0 \to \ldots$$

By the regularity assumption, this map $\cdot x$ is injective so

$$H^i((x_0 \ldots x_i), M) = 0$$
$$H^{i+1}((x_0 \ldots x_i), M) = M/(x_0M + \ldots + x_iM)$$

The proof of the "if"-part was omitted in the lecture but is clear according to Franke.

<u>Fact 2.1.1</u> c) Let $0 \to M' \to M \to M'' \to 0$ be a s.e.s. of R-modules. If x is M'-regular and M''-regular, then it is M-regular.

d) If $(x_0 \ldots x_n)$ is M-regular and $(k_0 \ldots k_n)$ are exponents then $(x_0^{k_0} \ldots x_n^{k_n})$ is also M-regular

<u>Proof</u>: c) Use (b) and the l.e.s. of cohomology groups.

d) For $n=0$ this is obvious. For $n \geq 0$, the module $M/x_0^{k_0}M$ has a filtration by submodules $x_0^iM/x_0^{k_0}M$ for $0 \leq i \leq k_0$, with filtration quotients $x_0^iM/x_0^{i+1}M \cong M/x_0M$.

Using c) several times we see that $(x_1, \dots, x_n)$ is $(M/x_0^k M)$-regular.
So by the induction hypothesis, $(x_1^{k_1}, \dots, x_n^{k_n})$ is also $(M/x_0^k M)$-regular.
Also x_0^k is M-regular (composition of injective map $\cdot x_0$ is again injective), so by a) the statement follows.

Definition 2.1.3 Let R be a ring, M, N free (or projective) finitely generated R-modules.
A non-degenerate pairing $M \times N \longrightarrow R$ is an R-bilinear map s.t. $M \longrightarrow \mathrm{Hom}_R(N, R)$ or $N \to \mathrm{Hom}_R(M, R)$ is an iso.

Corollary 2.1.1: Let R be any ring, $S \triangleq R[X_0, \dots, X_n]$. The Koszul-complex $C^*(\underline{X}^\ell, S)$ is acyclic in cohomological degree $\neq n+1$, ~~the cohomology~~ where $\underline{X}^\ell = (X_0^\ell, \dots, X_n^\ell)$. Its cohomology in degree $n+1$ is a free R-module.
By
$$C^m(\underline{X}^\ell, S)_k = \{ f \in C^m(\underline{X}^\ell, S) \mid f(i_0, \dots, i_m) \in \underset{\overset{\|}{S_{k+\ell m}}}{R[X_0, \dots, X_n]_{k+\ell m}} \}$$
the Koszul-complex becomes a complex of graded S-modules and
(a) $H^{n+1}(\underline{X}^\ell, S)_k = 0$ when $k > -n-1$
(b) We have an isomorphism
$$(4) \qquad H^{n+1}(\underline{x}^\ell, S)_{-n-1} \xrightarrow{\ \tau\ } R$$
$$\tau(f \bmod B^{n+1})_\ell = \text{coefficient of } (x_0 \dots x_n)^{\ell-1} \text{ in } f(0, \dots, n)$$
(c) For $0 < k < \ell$ we have a non-degenerate pairing
$$(5) \qquad S_k \times H^{n+1}(\underline{X}^\ell, S)_{-n-1-k} \xrightarrow{\text{multiplic.}} H^{n+1}(\underline{X}^\ell, S)_{-n-1} \xrightarrow[\cong]{(4)} R.$$

Proof: The vanishing of $H^i(\underline{X}^\ell, S)$ for $i \neq n+1$ follows from the fact that $\underline{X}^\ell$ is an S-regular sequence (example 2.1.1 and fact 2.1.1) together with fact 2.1.1. It also follows that $H^{n+1}(\underline{X}, S) \cong R/\langle X_0^\ell, \dots, X_n^\ell \rangle_S$, which is a free R-module with base $X^\alpha + \langle X_0^\ell, \dots, X_n^\ell \rangle_S$ where $\alpha = (\alpha_0, \dots, \alpha_n) \in \mathbb{N}^{n+1}$ satisfies $\alpha_i < \ell$ for $0 \leq i \leq n$. The elements of this base are homogeneous for the above grading and to get an element of homogeneous degree $-n-1$ of $H^{n+1}(\underline{X}, S)$ we need $|\alpha| = -n-1 + \ell(n+1) = (\ell-1)(n+1)$ for which $\alpha_0 = \dots = \alpha_n = \ell-1$ proving (b). To get elements of $H^{n+1}(\underline{X}, S)_k$ with $k > -n-1$ we would need $|\alpha| = k + \ell(n+1) > (\ell-1)(n+1)$ which is impossible as it forces at least one of the $n+1$ numbers α_i to be bigger than $\ell-1$, thus $\alpha_i \geq \ell$ proving (a).

The pairing from (c) pairs $X^\beta \in S_k$ with $(X^\alpha \bmod B^{n+1}) \in H^{n+1}(X, S)$ iff $\alpha + \beta = (\ell-1, \ell-1, \dots, \ell-1)$. For every α contributing to our base of $H^{n+1}(X, S)_{-n-1-k}$, the corresponding $\beta = (\ell-1, \dots, \ell-1) - \alpha$ is in $\mathbb{N}^{n+1}$ and $|\beta| = k$. If $k < \ell$ and $|\beta| = k$, all components of β are $\leq k \leq \ell-1$, hence $\alpha = (\ell-1, \dots, \ell-1) - \beta \in \mathbb{N}^{n+1}$ contributes ~~base~~ to our base of $H^{n+1}(X^\ell, S)$, showing the non-degeneracy of ~~our~~ the pairing.

Lemma 2.1.1: Let R be a Noetherian local ring, M a f.g. R-module and $\underline{x} = (x_0, \dots, x_n)$ a sequence of elements of the maximal ideal m of R.
Then $\underline{x}$ is M-regular iff $H^i(\underline{x}, M) = 0$ when $i \neq n+1$.
In particular, reordering a regular sequence $\underline{x}$ with $x_i \in m$ will again produce a regular sequence.
(Not needed for this lecture.)

2.2 The scheme $\mathbb{P}^n_R$ and its cohomology

Let $R_\bullet$ be an $\mathbb{N}$-graded ring. As a set

$$\mathrm{Proj}(R_\bullet) = \{ \mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \text{ is homogeneous, } R_+ \not\subseteq \mathfrak{p} \}$$

The closed sets have the form $V(I)$ for $I \subseteq R$ a homogeneous ideal, where

$$V_{\mathrm{Proj}}(I) = V(I) = \{ \mathfrak{p} \in \mathrm{Spec}\, R \mid \mathfrak{p} \text{ homogeneous}, I \subseteq \mathfrak{p} \},$$

which form a topology.
We define a sheaf of graded rings $\mathcal{O}(-)$ on $\mathrm{Proj}(R_\bullet)$ as follows:

$$\mathcal{O}(k)_{[\mathfrak{p}]} = (R_{\mathfrak{p}})_k$$

where $R_{\mathfrak{p}}$ is obtained from $R_\bullet$ by inverting all homogeneous elements of $R_\bullet \setminus \mathfrak{p}$.
Note that the subsets of the form $\mathrm{Proj}(R_\bullet) \setminus V(f)$ form a topology base (with $f \in R_\bullet$ homogeneous).
We put

$$(\mathcal{O}(k))(U) = \{ (g_{\mathfrak{p}}) \in \prod_{\mathfrak{p} \in U} (R_{\mathfrak{p}})_k \mid \text{coh. condition} \},$$

where the coh. cond. is that every $x \in U$ has an open nbhd $V = \mathrm{Proj}(R_\bullet) \setminus V(f) \subseteq U$, f homogeneous, s.t. there is $\sigma \in (R_f)_k$ s.t. $g_{\mathfrak{p}}$ is the image of σ under $R_f \to R_{\mathfrak{p}}$ for all $\mathfrak{p} \in U$.
We have operations

$$\mathcal{O}(k)(U) \times \mathcal{O}(k)(U) \longrightarrow \mathcal{O}(k)(U)$$
$$(f_{\mathfrak{p}}) + (g_{\mathfrak{p}}) \longmapsto (f_{\mathfrak{p}} + g_{\mathfrak{p}})_{\mathfrak{p} \in U}.$$

and

$$\mathcal{O}(k)(U) \times \mathcal{O}(\ell)(U) \longrightarrow \mathcal{O}(k+\ell)(U)$$
$$(f_{\mathfrak{p}}) \times (g_{\mathfrak{p}}) \longmapsto (f_{\mathfrak{p}}\, g_{\mathfrak{p}}).$$

and a map

$$\mathcal{O}(k)_{\mathfrak{p}} \longrightarrow \mathcal{O}(k)_{[\mathfrak{p}]}$$
$$((g_{\mathfrak{q}})_{\mathfrak{q}\in V}, V)/\sim \;\longmapsto\; g_{\mathfrak{p}}.$$

Fact 2.2.1: This map is an isomorphism.

Proposition 2.2.1: a) $\mathrm{Proj}(R.)$ is a scheme.
Open subsets of the form ~~R~~ ~~Pref~~ $\mathrm{Proj}(R.)\setminus V(f)$ for $f\in R_k$ with $k>0$ are affine and form a topology base.

b) Every finite subset of $\mathrm{Proj}(R.)$ is contained in an affine open subset.

c) The ~~sche~~ sheaf of modules $\mathcal{O}(n)$ on $\mathrm{Proj}(R)$ is q.c. and

$$\mathrm{Proj}(R.)\setminus V(f) \xrightarrow{\cong} \mathrm{Spec}(R_f)_0$$
$$\mathfrak{p} \longmapsto (\mathfrak{p}R_f)\cap(R_f)_0$$
$$\mathfrak{p}=\sqrt{(\mathfrak{q}R_f)}\cap R. \longleftarrow\!\!\shortmid\; \mathfrak{q}\in\mathrm{Spec}(R_f)_0$$

Lemma: Let Λ be a $\mathbb{Z}$-graded ring s.t. there are $k\in\mathbb{Z}\setminus\{0\}$ s.t. $\Lambda_k\cap\Lambda^* \neq \emptyset$. Then

$$\left(\begin{matrix}\text{homogeneous prime}\\ \text{ideals of } \Lambda\end{matrix}\right) \xleftrightarrow{\cong} \mathrm{Spec}\,\Lambda_0$$
$$\mathfrak{p} \longmapsto \mathfrak{q}=\mathfrak{p}\cap\Lambda_0$$
$$\mathfrak{p}=\sqrt{\mathfrak{q}\Lambda} \longleftarrow\!\!\shortmid\; \mathfrak{q}$$

Continuing proposition 2.2.1: This map in c) is a homeomorphism and $(R_{\mathfrak{p}})_0 \cong ((R_f)_0)_{\mathfrak{q}}$, identifying $\mathcal{O}(0)|_{\mathrm{Proj}(R)\setminus V(f)}$ with $\mathcal{O}_{\mathrm{Spec}(R_f)_0}$.

Moreover, we have similar isomorphism
$$(R_{\mathfrak{p}})_k \cong ((R_f)_k)_{\mathfrak{q}}$$
identifying
$$(2)\quad \mathcal{O}(k)|_{\mathrm{Proj}(R.)\setminus V(f)} \cong \widetilde{(R_f)_k}$$

d) When R is generated by R_1 as an R_0-algebra (or, equiv. R_+ by R_1 as an ideal in R) then the sheaves of modules $\mathcal{O}(k)$ are line bundles and the morphism
$$\mathcal{O}(k)\otimes_{\mathcal{O}_{\mathrm{Proj}}}\mathcal{O}(\ell) \longrightarrow \mathcal{O}(k+\ell) : f\otimes g \longmapsto fg$$

is an isomorphism

e) We have a morphism

$$(3) \quad \begin{array}{ccc} R\text{-}\mathrm{Proj}(R_\cdot) & \xrightarrow{\pi} & \mathrm{Spec}(R_0) \\ \mathfrak{p} & \longmapsto & \mathfrak{p} \cap R_0 \end{array}$$

and sending a section $\frac{r}{f^k} \in \mathcal{O}_{\mathrm{Spec}\, R_0}(\mathrm{Spec}\, R_0 \setminus V(f))$ to the section of $\mathcal{O}(0)$ on $\mathrm{Proj}(R_\cdot) \setminus V(f)$ defined by the same fraction. This is of finite type iff $R_\cdot$ is an R_0-algebra of finite type and

$$\pi^{-1}(\mathrm{Spec}\, R_0 \setminus V(f)) = \mathrm{Proj}(R_\cdot) \setminus V(f) \qquad \text{for } f \in R_0.$$

f) If $R_\cdot$ is Noetherian (equiv R_0 Noetherian, R/R_0 finite), then $\mathrm{Proj}(R_\cdot)$ is Noetherian and the $\mathcal{O}(k)$ are coherent.

<u>Remark 2.2.2</u>: For a graded $R_\cdot$-module $M_\cdot$ and open $U \subseteq \mathrm{Proj}(R_\cdot)$, let

$$(\widehat{M_\cdot})(U) = \{ (m_\mathfrak{q}) \in \prod_{\mathfrak{q} \in U} (M_\mathfrak{q})_0 \mid \text{locally image of } m \in (M_f)_0 \text{ for } f \notin \mathfrak{q} \text{ homogeneous} \}$$

defining a sheaf of modules on $\mathrm{Proj}(R_\cdot)$

On open subsets of the form $U = \mathrm{Proj}(R_\cdot) \setminus V(f)$ we have

$$\widehat{M}|_U \simeq \widetilde{(M_f)_0}, \quad \text{hence } \widehat{M} \text{ is quasi-compact.}$$

In the case where $M = R[k]$, then $\widehat{M} \simeq \mathcal{O}(k)$.

<u>Remark 2.2.3</u>: If $\varphi : R_\cdot \longrightarrow S_\cdot$ is a ring morphism s.t. $\varphi(R_k) \subseteq S_{dk}$ for some positive integer d and s.t. $S_+ = \sqrt{S\varphi(R_+)}$, then we have a continuous map

$$\begin{array}{ccc} \mathrm{Proj}(S_\cdot) & \longrightarrow & \mathrm{Proj}(R_\cdot) \\ \mathfrak{p} & \longmapsto & \varphi^{-1}(\mathfrak{p}) \end{array}$$

together with morphisms

$$(R_{\varphi^{-1}(\mathfrak{p})})_0 \xrightarrow{\varphi} (S_\mathfrak{p})_0$$

defining a morphism of preschemes $\mathrm{Proj}(S_\cdot) \longrightarrow \mathrm{Proj}(R_\cdot)$.

For instance, there is a morphism (a closed immersion) from $\mathbb{P}_A^m := \mathrm{Proj}(A[X_0, \dots, X_m])$ to $\mathbb{P}_A^n$ for $n \geq m$ defined by the ring morphism $A[X_0 \dots X_n] \longrightarrow A[X_0 \dots X_m]$ sending $X_{m+1}, \dots, X_n$ to 0.

By contrast, $A[X_0 \dots X_m] \subseteq A[X_0 \dots X_n]$ does <u>not</u> define a projection $\mathbb{P}_A^n \longrightarrow \mathbb{P}_A^m$ when $m < n$. As an example with $d > 1$, we take $R = A[X_0, X_1, X_2]$, $S = A[X, Y]$

and $\varphi: R \to S$ given by $\varphi(X_0) = X^2$, $\varphi(X_1) = XY$, $\varphi(X_2) = Y^2$.
The condition $S_+ = \sqrt{S.\varphi(R_+)}$ holds, as all monomials of even degree may be decomposed as a product of the three monomials on the r.h.s. We obtain a closed immersion
$\mathbb{P}^1_A = \mathrm{Proj}(S.) \longrightarrow \mathbb{P}^2_A = \mathrm{Proj}(R.)$, identifying $\mathbb{P}^1_A$ with the closed subscheme $V(X_0X_2 - X_1^2) \subseteq \mathbb{P}^2_A$.

<u>Remark</u>: Let (for a general ringed space X) the <u>tensor product</u> of two sheaves of R-modules M, N be given by <u>the sheafification</u> of

$$U \longmapsto M(U) \otimes_{R(U)} N(U),$$

in other words

$$(M \otimes_R N)(U) = \{ f \in \prod_{x \in U} M_x \otimes_{R_x} N_x \mid \text{condition} \}$$

where for all $y \in U$ there is open $V \subseteq U$ s.t. there is $\varphi \in M(V) \otimes_{R(V)} N(V)$ s.t. f_x is image of φ in $M_x \otimes_{R_x} N_x$.
This satisfies the universal property

$$\mathrm{Hom}(M \otimes_R N, T) \cong \mathrm{Bil}_R(M \times N, T)$$

On $X = \mathrm{Spec}(A)$, we have

$$\hat{M} \otimes_{\mathcal{O}_X} \hat{N} = \widehat{(M \otimes_A N)}$$

hence the tensor product of q.c. sheaves of modules over a q.c. sheaf of rings is again q.c.
When $\mathcal{L}$ is a line bundle, $M \otimes \mathcal{L}$ is locally (non-canonically) isomorphic to M:

$$M|_U \xrightarrow{\cong} (\mathcal{L} \otimes_{\mathcal{O}_X} M)|_U$$
$$m \longmapsto \lambda.m$$

where λ is a choice of a free generator on $\mathcal{L}|_U$.
Similarly, when $\mathcal{V}$ is an n-dimensional vector bundle, we locally have $M \otimes \mathcal{V} \cong M^n$. For $X = \mathrm{Proj}(R.)$, ~~[illegible]~~ for a graded ring $R.$ satisfying $R_+ = \langle R_1 \rangle$, we have isomorphisms

$$\widehat{M.[k]} \cong \hat{M.} \otimes \mathcal{O}(k)$$

$$(M_q)_k = (M[k]_q)_0 \ni m.p^k \longleftarrow m \otimes p^k$$

(where $m \in (M_0)_q)_0$, $p \in R_1 \setminus q$.)
In particular ($M = R[\ell]$) this proves our claim

$$\mathcal{O}(k) \otimes \mathcal{O}(\ell) \xrightarrow{\cong} \mathcal{O}(k+\ell)$$

of proposition 2.2.1.

Theorem 2: Let A be a ring, $X := \mathbb{P}^n_A := \mathrm{Proj}(R_\cdot)$ where $R_\cdot = A[X_0, \dots X_n]$ with the usual grading.

a) For $k \geq 0$ we have an isomorphism

$$(5) \qquad \begin{aligned} R_k &\xrightarrow{\cong} (\mathcal{O}(k))(X) \\ r &\longmapsto (\text{image of } r \text{ in } (R_\mathfrak{p})_k)_{\mathfrak{p} \in X} \end{aligned}$$

b) For $0 < p < n$, $H^p(X, \mathcal{O}(k)) = 0$ for all integers k.

c) For $n > 0$, $k > -n-1$ we have

$$H^n(\mathbb{P}^n_A, \mathcal{O}(k)) = 0$$

and for $k < 0$

$$H^0(\mathbb{P}^n_A, \mathcal{O}(k)) = 0.$$

d) There is an isomorphism $H^n(X, \mathcal{O}(-n-1)) \cong A$ defined by letting $\mathfrak{U}$ be the affine open cover $X = \bigcup_{i=0}^n \mathfrak{U}_i$ with $\mathfrak{U}_i = X \setminus V(X_i) = \mathrm{Spec}(A[X_0 \dots X_n][X_i^{-1}])$.

$$(6) \qquad \begin{aligned} {}^a\check{C}^n(\mathfrak{U}, \mathcal{O}(-n-1)) \cong A[X_0 \dots X_n]\left[\tfrac{1}{X_0 \cdots X_n}\right]_{-n-1} &\longrightarrow A \\ f = \sum_{\alpha \in \mathbb{Z}^n} f_\alpha X^\alpha &\longmapsto f_{(-1,-1,\dots,-1)} \end{aligned}$$

e) For all k, $H^0(X, \mathcal{O}(k))$ and $H^n(X, \mathcal{O}(k))$ are finitely generated free A-modules, and the pairing

$$(7) \qquad \begin{aligned} H^0(X, \mathcal{O}(k)) \times H^n(X, \mathcal{O}(-n-1-k)) &\longrightarrow H^n(X, \mathcal{O}(-n-1)) \cong A \\ (f, g) &\longmapsto f \cdot g \\ \mathcal{O}(k)(X) \ni \quad {}^a\check{C}^n(\mathfrak{U}, \mathcal{O}(-n-1-k)) \quad & \qquad {}^a\check{C}^n(\mathfrak{U}, \mathcal{O}(-n-1)) \end{aligned}$$

is non-degenerate.

Remark: The pairing (7) with values in $H^n(X, \mathcal{O}(-n-1))$ is canonical, while the isomorphism $H^n(X, \mathcal{O}(-n-1)) \cong A$ is not.

Proof: When $n = 0$, $X = \mathrm{Spec}(A)$ and $\mathcal{O}(k) \cong \mathcal{O}_X \cdot X_0^k$, and ~~overt~~ everything follows from basic facts about affine preschemes. Thus let $n > 0$. Let $\mathfrak{U}: X = \bigcup_{i=1}^n \mathfrak{U}_i$ as in (6). We have

$$\begin{aligned} \mathcal{O}(k)(\mathfrak{U}_{i_0 \dots i_\ell}) &\cong \mathcal{O}(k)(X \setminus V(X_{i_0} \cdots X_{i_\ell})) \\ &\cong \mathcal{O}(k)\left(\mathrm{Spec}\left(A[X_0 \dots X_n]\left[\tfrac{1}{X_{i_0} \cdots X_{i_\ell}}\right]\right)_0\right) \\ &\cong A[X_0 \dots X_n]\left[\tfrac{1}{X_{i_0} \cdots X_{i_\ell}}\right]_k \end{aligned}$$

Thus

$$\begin{aligned} \coprod_{k=-\infty}^{\infty} \mathcal{O}(k)(\mathfrak{U}_{i_0 \dots i_\ell}) &\cong A[X_0 \dots X_n]\left[\tfrac{1}{X_{i_0} \cdots X_{i_\ell}}\right] \\ &= \bigcup_{e=0}^{\infty} A[X_0 \dots X_n] \cdot \frac{1}{(X_{i_0} \cdots X_{i_\ell})^e} \end{aligned}$$

which is canonically isomorphic to the direct limit

$$\varinjlim \left(\underset{0}{R} \xrightarrow{X_{i_0}\cdots X_{i_\ell}} \underset{1}{R} \xrightarrow{X_{i_0}\cdots X_{i_\ell}} \underset{2}{R} \longrightarrow \underset{3}{\cdots\cdots} \right).$$

where the isomorphism sends an element γ of the eth sequence number to $\frac{\gamma}{(X_{i_0}\cdots X_{i_\ell})^e}$

These assemble to an isomorphism

$$\coprod_{k=-\infty}^{\infty} \check{C}^{a\vee\ell}(\mathcal{U}, \mathcal{O}(k)) \xleftarrow{\cong} \varinjlim_{e} C^{\ell+1}(\underline{X}^e, R),$$

where the transition morphism $C^m(\underline{X}^e, R) \longrightarrow C^m(\underline{X}^{e+1}, R)$ sends $f \in C^m(\underline{X}^e, R)$ to φ defined by

$$\varphi(i_1 \ldots i_m) = X_{i_1}\cdots X_{i_m} f(i_1, \ldots, i_m)$$

The complex

$$R_\cdot \longrightarrow \coprod_{k=-\infty}^{\infty} \check{C}^{\vee 0}(\mathcal{U}, \mathcal{O}(k)) \xrightarrow{d} \coprod_{k=-\infty}^{\infty} \check{C}^{\vee 1}(\mathcal{U}, \mathcal{O}(k)) \longrightarrow \cdots$$

is thus identified with

$$\varinjlim_{e} \check{C}^{\vee *}(\underline{X}^e, R_\cdot)[-1]$$

where the grading is identified with the grading in proposition 2.2.1

For $n \geq 1$

$$H^0(X, \mathcal{O}(k)) \cong R_k \qquad \text{(canonically)}$$

$$H^n(X, \mathcal{O}(n-1)) \cong A \qquad \text{(non-canonically)}$$

We have a non-degenerate pairing

$$H^0(X, \mathcal{O}(k)) \times H^n(X, \mathcal{O}(-n-1-k)) \longrightarrow H^n(X, \mathcal{O}(-n-1)) \cong A$$

Also $H^p(X, \mathcal{O}(-n-1)) = 0$ for $0 < p < n$.

Our next aim is to investigate the trivialization of the A-module $H^n(\mathbb{P}^n_A, \mathcal{O}(-n-1))$, which is non-canonical unless the coordinates and their orders are fixed. When the coordinates are changed by an element of $GL_{n+1}(A)$ (inducing an automorphism of R, hence of $\mathrm{Proj}(R_\cdot) = \mathbb{P}^n_A$) with a compatible action on the line bundle $\mathcal{O}(k)$.

<u>Lemma 2.2.1</u> $g \in GL_{n+1}(A)$ acts on $H^n(\mathbb{P}^n_A, \mathcal{O}(-n-1))$ by multiplication by $\det(g)^{-1} \in A^*$

<u>Proof</u>: Let $\varphi_A : GL_{n+1}(A) \longrightarrow A^*$ be the homomorphism under investigation. If $\varphi : A \to B$ is a ring morphism, we have a

we have a graded morphism $A[X_0 \ldots X_n] \to B[X_0 \ldots X_n]$ that defines a morphism $Proj(B) \xrightarrow{p} Proj(A)$ with a morphism

$$\mathcal{O}(k)_{\mathbb{P}^n_A} \longrightarrow p_* \mathcal{O}(k)_{\mathbb{P}^n_B}$$

defined by the above morphism on open subsets of the form $\mathbb{P}^n \setminus V(X_i)$

By the functorial property of cohomology investigated in 1.3 we have a commutative diagram

$$\begin{array}{ccc} H^n(\mathbb{P}^n_A, \mathcal{O}(-n-1)) & \xrightarrow{p^*} & H^n(\mathbb{P}^n_B, \mathcal{O}(-n-1)) \\ \downarrow g & & \downarrow \alpha(g) \\ H^n(\mathbb{P}^n_A, \mathcal{O}(-n-1)) & \xrightarrow{p^*} & H^n(\mathbb{P}^n_B, \mathcal{O}(-n-1)) \end{array}$$

showing that $\varphi_B(\alpha(g)) = \alpha(\varphi_A(g))$.

In general, there is a ring homomorphism

$$B = \mathbb{Z}[Y_{ij}|_{i,j=0}^n]\left[\frac{1}{\det(Y_{ij})_{i,j=1}^n}\right] \longrightarrow A$$

sending Y_{ij} to the corresponding matrix element of g and $\frac{1}{\det(Y_{ij})}$ to $\frac{1}{\det(g)}$, reducing the assertion to the case of the infinite domain B.

Applying a similar argument to the ring morphism of B to an algebraic closure of the fraction field of B we may assume A to be an algebraically closed field.

In this case, it is known from (advanced) linear algebra that the commutator group of $GL_{n+1}(A)$ is $SL_{n+1}(A)$.

A being an algebraically closed field, every element of $GL_{n+1}(A)$ is a product of the form $\begin{pmatrix} a & & 0 \\ & \ddots & \\ 0 & & a \end{pmatrix}$ and an element of $SL_{n+1}(A)$. This reduces us to the case of the matrices of the matrices $\begin{pmatrix} a & & \\ & \ddots & \\ & & a \end{pmatrix}$ which acts as the identity on $Proj(R_\cdot)$ and as multiplication by a^k on $\mathcal{O}(k)$, proving our claim.

<u>Definition 2.2.1</u>: A line bundle $\mathcal{L}$ on a quasi-compact prescheme X is called <u>ample</u> if for every locally finitely generated q.c. $\mathcal{O}_X$-module M and sufficiently small (negative) integers m, there is an epimorphism

$$(\mathcal{L}^{\otimes m})^k \longrightarrow M$$

for some natural number k.

Remark a) We have put $\mathcal{L}^{-1} = \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$ and

$$\mathcal{L}^{\otimes -m} \cong (\mathcal{L}^{-1})^{\otimes m} \quad \text{for} \quad m \in \mathbb{N}$$

These are again line-bundles, because in the case $\mathcal{L} = \mathcal{O}_X$ they are canonically isomorphic to $\mathcal{O}_X$, which locally up to isomorphism is always the case.

For the same reason, the morphism $\mathcal{L} \otimes \mathcal{L}^{-1} \xrightarrow{\cong} \mathcal{O}_X$ defined by the bilinear map of evaluation

$$\begin{aligned} \mathcal{L} \times \mathcal{H}om(\mathcal{L}, \mathcal{O}_X) &\longrightarrow \mathcal{O}_X \\ (\lambda, \eta) &\longmapsto \eta(\lambda) \end{aligned}$$

is an isomorphism.

~~That~~ More generally,

$$\mathcal{L} \otimes \mathcal{H}om(\mathcal{L}, \mathcal{M}) \xrightarrow{\cong} \mathcal{M}$$

for line bundles and isomorphism classes of line bundles get an <u>abelian group structure</u>, with $\otimes$ defining multiplication, $\mathcal{O}_X$ being the identity and $\mathcal{L} \mapsto \mathcal{L}^{-1}$ being the inverse.

b) On $\mathbb{P}^n_A$,

$$\begin{aligned} \mathcal{O}(k) &\cong \mathcal{H}om(\mathcal{O}(\ell), \mathcal{O}(\ell + k)) \\ f &\longmapsto (g \mapsto f g), \end{aligned}$$

showing in particular that $\mathcal{O}(k)^{-1} = \mathcal{O}(-k)$

c) The definition could also be replaced by the condition that $\mathcal{L}^{\otimes k} \otimes \mathcal{M}$ is generated by its global sections when k is large enough.

d) When X is affine, $\mathcal{O}_X$ is ample.

e) The definition 2.2.1 must not be confused with a <u>relative</u> notion of ampleness.

<u>Definition 2.2.2</u>: Let $\mathcal{L}$ be a line-bundle on a locally ringed space X, let $U \subseteq X$ be open and let $\lambda \in \mathcal{L}(U)$. Define

$$V(\lambda) = \{x \in X \mid \text{the image of } \lambda \text{ in } \mathcal{L}_x \text{ is in } \mathfrak{m}_x \mathcal{L}_x\}$$

<u>Remark</u>: a) When $\mathcal{L} = \mathcal{O}_X$, this coincides with the definition we gave before, in which case $V(\lambda)$ is closed. Since $\mathcal{L}$ is locally isomorphic to $\mathcal{O}_X$ and closedness can be checked locally, $V(\lambda)$ is closed in general.

b) For line bundles $\mathcal{L}, \mathcal{M}$ for a section $\lambda \otimes \mu$ on $\mathcal{L} \otimes \mathcal{M}$ we get

$$V(\lambda \otimes \mu) = V(\lambda) \cup V(\mu).$$

<u>Lemma 2.2.2</u>: Let X be a q.c. prescheme and $\mathcal{L}$ a line bundle on X with global section $\lambda \in \mathcal{L}(X)$. Let $\mathcal{M}$ be a q.c. $\mathcal{O}_X$-module.

a) If $m \in \mathcal{M}(X)$ with $m|_{X \setminus V(\lambda)} = 0$, there is $k \in \mathbb{N}$ s.t. the section $\lambda^{\otimes k} \otimes m = 0$ in $(\mathcal{L}^{\otimes k} \otimes \mathcal{M})(X)$.

b) If X is in addition quasi-separated and $m \in \mathcal{M}(X \setminus V(\lambda))$, then there is $k \in \mathbb{N}$ st. $\lambda^{\otimes k} \otimes m \in (\mathcal{L}^{\otimes k} \otimes \mathcal{M})(X \setminus V(\lambda))$ extends to a global section of $\mathcal{L}^{\otimes k} \otimes \mathcal{M}$ on X.

<u>Remark</u>: When $\mathcal{L} = \mathcal{O}_X$, this is identical to one of our definitions of q.c.-ness for sheaves of modules.

<u>Proof</u>: When $\mathcal{L} = \mathcal{O}_X$, a) is equivalent to the injectivity and b) to the surjectivity of

$$\mathcal{M}(X)_\lambda \longrightarrow \mathcal{M}(X \setminus V(\lambda)).$$

Since this holds when $\mathcal{M}$ is q.c. and X is q.c. (resp. q.c. and q.s.) the assertion holds when $\mathcal{L}$ is trivial.

As X is q.c., we have $X = \bigcup_{i=1}^{n} U_i$ where $U_i \subseteq X$ is open and q.c. (e.g. affine) s.t. $\mathcal{L}|_{U_i}$ is trivial.

For a) we have $(m|_{U_i})|_{U_i \setminus V(\lambda)} = 0$, hence by the special case just treated there are $k_i \in \mathbb{N}$ s.t. $\lambda^{k_i} \otimes m = 0$ on U_i, so for $k = \max(k_1, \dots, k_n)$ $\lambda^{\otimes k} \otimes m = 0$ on X.

For b) there is $\ell \in \mathbb{N}$ s.t. $\lambda^{\ell} \otimes m = \mu_i|_{U_i}$ for some $\mu_i \in (\mathcal{L}^{\otimes \ell} \otimes \mathcal{M})(U_i)$. As X is q.s., $U_i \cap U_j$ is q.c., hence by a) and $\mu_i|_{U_i \cap U_j \setminus V(\lambda)} = \mu_j|_{U_i \cap U_j \setminus V(\lambda)}$, there is $a \in \mathbb{N}$ s.t. $\lambda^a \cdot \mu_i = \lambda^a \mu_j$ on $U_i \cap U_j$. Thus there is $\mu \in (\mathcal{L}^{\otimes k} \otimes \mathcal{M})(X)$ with $k = a + \ell$ st.

$$\mu|_{U_i} = \lambda^a \mu_i.$$

Then $\mu_{X \setminus V(\lambda)} = \lambda^k \otimes m$, as this holds on the $U_i \setminus V(\lambda)$.

<u>Proposition 2.2.2</u>: Let X be a q.c. prescheme and $\mathcal{L}$ a line bundle on X s.t. X may be covered by affine open subsets of the form $X \setminus V(\lambda)$, where $\lambda \in \mathcal{L}(X)$. Then $\mathcal{L}$ is ample.

Corollary 2.2.1 a) $\mathcal{O}(1)$ on $\mathbb{P}^n_A$ is ample.
(The $U_i = \mathbb{P}^n \setminus V(X_i)$ are affine and cover $\mathbb{P}^n_A$)
b) If X is quasi-affine, then $\mathcal{O}_X$ is ample.
(e.g. $X = \bar{X} \setminus Z$ with $\bar{X}$ affine, Z closed. Then the open subsets of the form $X \setminus V(f|_X) = \bar{X} \setminus V(f)$ with $f \in \mathcal{O}_{\bar{X}}(\bar{X})$ s.t. $V(f) \supseteq Z$ are affine and cover X.)

Proof: Let $X = \bigcup_{i=1}^n U_i$ with $U_i = X \setminus V(s_i)$, affine, $s_i \in \mathcal{L}(X)$.
Let M be a locally f.g. q.c. $\mathcal{O}_X$-module. Then $M_i := M(U_i)$ is f.g. over $\mathcal{O}_X(U_i)$, say by $(m_{ij})_{j=1}^K$. Replacing M by $M \otimes \mathcal{L}^{\otimes k}$ and m_{ij} by $m_{ij} \otimes s_i^{\otimes k}$, we achieve by lemma 2.2.2 that m_{ij} extends to a section of M on X which we denote by m_{ij} too. Then the m_{ij} generate M. $\square$

Theorem 3 (Serre): Let A be a Noetherian ring. If M is a coherent sheaf of modules on $X = \mathbb{P}^n_A$, then the groups $H^i(X, M)$ are f.g. A-modules.
When p is positive and k sufficiently large,
$$H^p(X, M(k)) = 0$$
where $M(k) := M \otimes \mathcal{O}_X(k)$.

~~Since~~
Proof: Since $H^p(X, M) = \check{H}(\mathfrak{U}, M)$ where $\mathfrak{U}$ is the covering used in the proof of theorem 2, $H^p(X, M) = 0$ when $p > n$.
So for $p > n$ the assertion is trivial.
We prove the assertion by downward induction on p. As $\mathcal{O}(1)$ is ample there is an epimorphism $\mathcal{O}(-k)^{\oplus l} \to M$ for k sufficiently large. Hence we have a s.e.s.
$$0 \to K \to \mathcal{O}(-k)^{\oplus l} \to M \to 0$$
and thus
$$\cdots \to H^p(X, \mathcal{O}(-k)^{\oplus l}) \to H^p(X, M) \to H^{p+1}(X, K) \to \cdots$$
$$H^p(X, \mathcal{O}(-k)^{\oplus l}) \cong H^p(X, \mathcal{O}(-k))^{\oplus l}$$
As A is Noetherian, K is coherent and the right end is f.g. over A by the induction assumption.
For the left end, this holds by theorem 2. So the term in the middle is f.g.
Similarly $H^p(X, \mathcal{O}(m-k))^{\oplus l} \to H^p(X, M(m)) \to H^{p+1}(X, K(m))$
If $m \geq k-n$ and $p > 0$, the left term vanishes by

theorem 2. The same holds for the right side when m is large enough, by the induction assumption.

Proposition 2.2.3: Let R be a Noetherian ring and X a closed subprescheme of $\mathbb{P}^n_R$. For sufficiently large k the map

$$H^0(\mathbb{P}^n_R, \mathcal{O}(k)) = R[X_0, \dots, X_n]_k \longrightarrow H^0(X, \mathcal{O}(k))$$

given by restriction, is surjective.

Proof: Let $\mathcal{J} \subseteq \mathcal{O}_{\mathbb{P}^n_R}$ be the sheaf of ideals defining $X \subseteq \mathbb{P}^n_R =: Y$. We have a s.e.s.

$$0 \longrightarrow \mathcal{J}(k) \longrightarrow \mathcal{O}_Y(k) \longrightarrow i_* \mathcal{O}_X(k) \longrightarrow 0,$$

where $i: X \longrightarrow Y$ is the embedding,
~~Since i is a closed immersion ...~~
giving us

$$\begin{array}{ccccc} H^0(Y, \mathcal{O}_Y(k)) & \longrightarrow & H^0(X, \mathcal{O}_X(k)) & \longrightarrow & H^1(Y, \mathcal{J}(k)) \\ \cong & & \| & & \\ R[X_0, \dots, X_n]_k & & H^0(Y, i_* \mathcal{O}_X(k)) & & \end{array}$$

and by Theorem 3, the right term vanishes for k large enough. □

Remark: We have used

$$i_*(\mathcal{M}) \otimes \mathcal{L} \cong i_*(\mathcal{M} \otimes i^* \mathcal{L})$$

which holds even when $\mathcal{L}$ is not a line bundle but in that case holds trivially.

~~Exercise~~
Example 2.2.2: Let k be a field, $n \geq 2$ and Z a closed reduced subprescheme of $\mathbb{P}^n_k$ whose irreducible components are of codimension 1 (e.g. a curve in $\mathbb{P}^2_k$.)
Then $k[X_0, \dots, X_n]_\ell \longrightarrow H^0(Z, \mathcal{O}_Z(\ell))$ is surjective.

Proposition 2.2.4 (Serre): Let A be a Noetherian ring, M a graded module over $R = A[X_0, \dots, X_n]$ and $\mathcal{M} = \tilde{M}$ on $\mathbb{P}^n_A$. We assume M to be f.g., so that $\mathcal{M}$ is coherent.

a) When k is sufficiently large, $M_k \longrightarrow H^0(\mathbb{P}^n_A, \mathcal{M}(k)) \cong (\tilde{M}(\ell))(\mathbb{P}^n_A)$ is surjective.

b) When $A = k$ is a field, the characteristic

$$\chi(\mathbb{P}^n_k, \mathcal{M}(\ell)) := \sum_{p=0}^n (-1)^p \dim_k(H^p(X, \mathcal{M}(\ell)))$$

(which for sufficiently ~~high~~ large ℓ is just $\dim_k H^0(X, \mathcal{M}(\ell))$) is equal to $P_M(\ell)$, the Hilbert polynomial of M.

Proof a) When $m \in M_k$ is in the kernel, it is an element of $N. := \bigcup_{\ell=0}^{\infty} \{ m \in M. \mid X_i^\ell m = 0$ for some $i \} \subseteq M.$

As $R. = A[X_0, \dots, X_n]$ is Noetherian, N is f.g. and hence there is ℓ s.t. X_i^ℓ annihilate all of N.

Then N vanishes in degrees $> d + (n+1)\ell$, where d is the maximum degree of the chosen generators for M, and in such degrees the map is surjective.

For $M = R$ and $k \geq 0$ (or $n > 0$) we have $\mathcal{M} = \mathcal{O}_{\mathbb{P}^n_A}$, and the surjectivity holds by theorem 2.

In general, we have a surjection $\bigoplus_{i=1}^{N} R[d_i] \longrightarrow M$ with kernel K, giving a s.e.s.

$$0 \longrightarrow \mathcal{K} \longrightarrow \bigoplus_{i=1}^{N} \mathcal{O}_{\mathbb{P}^n_A}(d_i) \longrightarrow \mathcal{M} \longrightarrow 0.$$

When ℓ is so large that $H^1(\mathbb{P}^n_A, \mathcal{K}(\ell)) = 0$, then every every element of $H^0(\mathbb{P}^n_A, \mathcal{M}(\ell))$ comes from an element of $\bigoplus_{i=1}^{N} (\mathcal{O}_{\mathbb{P}^n_A}(d_i + \ell))(\mathbb{P}^n_A) = \bigoplus_{i=1}^{N} R_{d_i+\ell}$.

(The equality holding when $n > 0$ or all $\ell + d_i \geq 0$), hence from an element of M_ℓ.

b) By a), Theorem 3 and the theory of the Hilbert polynomial, the assertion holds for sufficiently large ℓ. Hence it is sufficient to show that

$$\ell \longmapsto \chi(\mathbb{P}^n_A, \mathcal{M}(\ell))$$

is always given by a polynomial in ℓ.

We show this by induction on n, the case $n=0$ being trivial. By the additivity of the Euler-poincaré characteristic, if the assertion is true for two of the three members of the ses $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$, then it is valid for the third one.

The exact sequence

$$0 \longrightarrow K \longrightarrow M \xrightarrow{X_n} M(1) \longrightarrow Q \longrightarrow 0$$

gives

$$0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$$

and

$$0 \longrightarrow N \longrightarrow M(1) \longrightarrow Q \longrightarrow 0$$

So $\chi(\mathcal{M}(\ell)) = \chi(\mathcal{K}(\ell)) + \chi(\mathcal{N}(\ell)) = \chi(\mathcal{K}(\ell)) + \chi(\mathcal{M}(\ell+1)) - \chi(\mathcal{Q}(\ell))$. Hence

Hence

$$\chi(M(\ell+1))-\chi(M(\ell)) = \chi(Q(\ell)) - \chi(K(\ell))$$

But the r.h.s. of this equation defines a polynomial in ℓ as K and Q are annihilated by X_n, and thus are direct images under $\mathbb{P}^{n-1}_A \longrightarrow \mathbb{P}^n_A$ of coherent sheaves of modules on $\mathbb{P}^{n-1}_A$ to which the induction hypothesis applies.

This implies that $\chi(M(\ell))$ is also a polynomial, and thus we are done. $\square$

Remark. (Line bundles and Weyl divisors on locally factorial Noetherian schemes.)

Let R be a Noetherian local normal domain (i.e. ~~loc.~~ int. closed in quotient field K). Then

$$(*) \qquad R = \bigcap_{ht(p)=1} R_p,$$

where the R_p are DVR's.

Now let X be Noetherian, normal and connected, with generic point η, and $K = \mathcal{O}_{X,\eta}$. It defines a q.c. sheaf of modules $\mathcal{K}$ by

in particular integral so irreducible

$$\mathcal{K}(U) = \begin{cases} 0 & U=\emptyset \\ K & \text{otherwise} \end{cases}$$

and there is a function

$$\mathrm{div}: \mathcal{K}(U)^* \longrightarrow \coprod_{x\in U\cap X_1} \mathbb{Z}$$

where $\mathrm{div}(f)_x = v_x(f)$, (where f is viewed as an element of K, which is also the field of quotients of $\mathcal{O}_{X,x}$ which is a DVR with valuation v_x.)

and

$$X_1 = \{x\in X \mid \mathrm{codim}(\overline{\{x\}}, X)=1\}$$

By $(*)$, we have

$$\mathcal{O}_X^*(U) = \mathrm{Ker}\left(\mathcal{K}^*(U) \xrightarrow{\mathrm{div}} \coprod_{x\in U\cap X_1} \mathbb{Z}\right)$$

Similarly

$$\mathcal{O}_X(U) = \{f\in \mathcal{K}(U) \mid \mathrm{div}(f)\geq 0\}$$

Let $\mathrm{Div}(X)$ be the set of all formal $\mathbb{Z}$-linear combinations of elements of X_1, i.e. $\mathrm{Div}(X) = \coprod_{x\in X_1}\mathbb{Z}$. For $D\in\mathrm{Div}(X)$, let $(\mathcal{O}_X(D))(U) = \{f\in\mathcal{K}(U) \mid \mathrm{div}(f)+D|_U \geq 0\}$

When X is locally factorial, this is a line bundle.
(In general, ~~this~~ for normal X, it is a ~~s~~ coherent sheaf of modules.)

__Lemma__ Assume all objects to be f.g. graded k-vector spaces.

a) $\sum_{i=-\infty}^{\infty} (-1)^i \dim_k H^i(C^\bullet) = \sum_{i=-\infty}^{\infty} (-1)^i \dim_k (C^i)$

b) For a l.e.s. $\longrightarrow C_{i-1} \longrightarrow A_i \longrightarrow B_i \longrightarrow C_i \longrightarrow A_{i+1} \longrightarrow \dots$

$$\sum_{i=-\infty}^{\infty} (-1)^i \dim B_i = \sum_{i=-\infty}^{\infty} (-1)^i (\dim A_i + \dim C_i)$$

2.3 __Projective morphisms__

There are several definitions of "projective morphisms" in common use. We will give the EGA-definition. The __Proj__ construction occurring in the definition is very important.

Let $R_\bullet$ be a graded q.c. $\mathcal{O}_X$-algebra, where X is a prescheme. All $(R_i)_{i=1}^{\infty}$ are q.c. $\mathcal{O}_X$-modules and there are $\mathcal{O}_X$-bilinear multiplications $R_i \times R_j \longrightarrow R_{i+j}$, which is commutative and assoc. and have unit in R_0.

We now construct __Proj__ $(R_\bullet)$. As a __set__

$$\underline{\mathrm{Proj}(R_\bullet)} = \{(x,\mathfrak{p}) \mid x \in X,\ \mathfrak{p} \in \mathrm{Spec}(R_\bullet)_x \text{ s.t. } \dots\}$$

where we take all tuples with $\mathfrak{p} \subseteq (R_\bullet)_x$ a homogeneous prime ideal not containing the augmentation ideal such that its preimage under $\mathcal{O}_{X,x} \longrightarrow (R_0)_x \longrightarrow (R_\bullet)_x$ is $\mathfrak{m}_{X,x}$.

Here we put $(R_\bullet)_x = \coprod_{i=0}^{\infty} (R_i)_x$.

The __topology__ of __Proj__ $(R_\bullet)$ is formed by subsets $\Omega(U,f)$, where $U \subseteq X$ is open, $f \in R_m(U)$ for some $m \in \mathbb{N}$.

$$\Omega(U,f) := \{(x,\mathfrak{p}) \mid x \in U,\ \text{image of } f \text{ in } (R_\bullet)_x \text{ not in } \mathfrak{p}\}$$

This is a topology base, as

$$\underline{\mathrm{Proj}}(R_\bullet) = \Omega(X,1)$$
$$\Omega(U,f) \cap \Omega(V,g) = \Omega(U \cap V, f\cdot g)$$

For the __structure sheaf__, we take the sheaffification of

$$\Omega(U,f) \longmapsto (R_\bullet(U)_f)_0,$$

where $(R_\bullet(U))_f = \coprod_{i=0}^{\infty} R_i(U)_f$

More explicitly, define

$$\mathcal{O}_{\mathrm{Proj}(R_\bullet),(x,\mathfrak{p})} = [(R_x)_{\mathfrak{p}}]_0$$

Then

$$\mathcal{O}_{\underline{\mathrm{Proj}}(R.)}(W) = \{ (\rho_{x,\mathfrak{p}}) \in \prod_{(x,\mathfrak{p})\in W} \mathcal{O}_{\underline{\mathrm{Proj}}(R.),(x,\mathfrak{p})} \mid \text{condition} \}$$

where for every $z \in W$ there should be an open neighborhood $\Omega(U, f) =: V$ of z in W and an element $r \in [(R.(U))_f]_0$ s.t.

$$\rho_{x,\mathfrak{p}} = (\text{image of } r \text{ under } R.(U)_f \to (R_x)_f \to (R_x)_{\mathfrak{p}}$$

holds for all $(x,\mathfrak{p}) \in V$.

It follows that $\underline{\mathrm{Proj}}(R.)$ is a locally ringed space.

There is a continuous map $\underline{\mathrm{Proj}}(R.) \xrightarrow{\pi} X$, which is continuous as $\pi^{-1}(U) = \Omega(U,1)$.

For the algebraic component $\pi^{\#}: \mathcal{O}_X \to \pi_* \mathcal{O}_{\underline{\mathrm{Proj}}(R.)}$ we simply map $f \in \mathcal{O}_X(U)$ to the collection of images under

$$\mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X,x} \longrightarrow (R_0)_x \longrightarrow [(R._x)_{\mathfrak{p}}]_0$$

for all $(x,\mathfrak{p}) \in \pi^{-1}(U)$.

So $\pi : \underline{\mathrm{Proj}}(R.) \longrightarrow X$ is a morphism of locally ringed spaces.

The construction is base-local:

$$\underline{\mathrm{Proj}}(R.|_U) \cong \pi^{-1}(U)$$

When $R.$ is a graded ring and $X = \mathrm{Spec}\, R_0$, then for $\mathcal{R}. = \widetilde{R.}$ we have

$$\underline{\mathrm{Proj}}(\mathcal{R}.) \cong \mathrm{Proj}(R.)$$

comparing the definitions. Since the following assertions are base-local they follow from the similar assertions about $\mathrm{Proj}(R.)$:

Fact 23.1 a) $\underline{\mathrm{Proj}}(\mathcal{R}.)$ is a prescheme.
In fact, for any affine open $U \subseteq X$, $\pi^{-1}(U)$ is a scheme.

b) π is separated. When $\mathcal{R}.$ is locally f.g. as an $\mathcal{O}_X$-algebra then π is of finite type and $\underline{\mathrm{Proj}}(\mathcal{R}.)$ is Noetherian (locally Noeth.) if X is Noeth. (locally Noeth.).

(c) When $\mathcal{R}.$ is generated by $\mathcal{R}_0$ and $\mathcal{R}_1$ as an $\mathcal{O}_X$-algebra, then the $(\mathcal{O}_{\underline{\mathrm{Proj}}\,\mathcal{R}})$-module $\mathcal{O}(\ell) := \mathcal{O}_{\underline{\mathrm{Proj}}\,\mathcal{R}}(\ell)$ obtained by changing 0 to ℓ in above definition is a line bundle.
Moreover, $\mathcal{O}(k)^{-1} \cong \mathcal{O}(-k)$ and $\mathcal{O}(k) \otimes \mathcal{O}(\ell) \xrightarrow{\cong} \mathcal{O}(k+\ell)$.
(tensor product over $\mathcal{O}_{\underline{\mathrm{Proj}}(R.)}$, morphism via $R_k \times R_\ell \to R_{k+\ell}$)

We have a canonical morphism

$$\pi^* \mathcal{R}_k \xrightarrow{\rho} \mathcal{O}(k)$$

sending $r \in \mathcal{R}_k(\mathcal{U})$ to its image under the images under

$$\mathcal{R}_k(\mathcal{U}) \longrightarrow [(\mathcal{R}_x)_\Box]_k \quad \text{where } (x,\Box) \text{ runs over } \pi^{-1}(\mathcal{U}).$$

Remark: The inverse image functors

$f^b : \mathrm{PreSh}(X) \longrightarrow \mathrm{PreSh}(Y)$ left adjoint to $f_* : PS(Y) \to PS(X)$

$f^\# : \mathrm{Sh}(X) \longrightarrow \mathrm{Sh}(Y)$ —— " —— $f_* : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$

$f^* : \mathcal{O}_X\text{-mod} \longrightarrow \mathcal{O}_Y\text{-mod}$ —— " —— $f_* : \mathcal{O}_Y\text{-mod} \to \mathcal{O}_X\text{-mod}$

where $Y \xrightarrow{f} X$ is continuous resp. a morphism of ringed spaces, are defined by

$$(f^b \mathcal{G})(\mathcal{U}) = \varinjlim_{V \supseteq f(\mathcal{U})} \mathcal{G}(V)$$

$$f^\# \mathcal{G} = \text{sheafification of } f^b \mathcal{G}.$$

Note that $(f^\# \mathcal{G})_x \xrightarrow{\simeq} \mathcal{G}_{f(x)}$

$$(f^b \mathcal{G})_x \simeq \mathcal{G}_{f(x)}$$

In fact

$$\mathcal{G}_x = (i_x^{b} \mathcal{G})(\{x\})$$

where $i_x : \{x\} \longrightarrow X$ is the embedding.

Also $(gf)^* \simeq f^* g^*$ canonically.

Moreover

$$f^* \mathcal{M} = f^\# \mathcal{M} \otimes_{f^\# \mathcal{O}_X} \mathcal{O}_Y.$$

which we can describe in more down-to-the-earth terms as

$$(f^* \mathcal{M})_y = \mathcal{M}_{f(y)} \otimes_{\mathcal{O}_{X,f(y)}} \mathcal{O}_{Y,y}$$

$$(f^* \mathcal{M})(\mathcal{U}) = \{ (\varphi_y) \in \prod_{y \in \mathcal{U}} (f^* \mathcal{M})_y \mid \text{locally from } \mathcal{M}(W) \otimes_{\mathcal{O}_X(W)} \mathcal{O}_Y(V) \}$$
$$f(V) \subseteq W$$

When $\mathcal{M} = \mathcal{O}_X^{\oplus n}$, then $f^* \mathcal{M} \simeq \mathcal{O}_Y^n$, e.g.

$$(f^* \mathcal{M})_y = \mathcal{O}_{X,f(y)}^n \otimes_{\mathcal{O}_{X,f(y)}} \mathcal{O}_{Y,y} \simeq \mathcal{O}_{Y,y}^n.$$

Lemma 23.1: Let $R_\bullet$ be a $\mathbb{N}$-graded ring generated by R_0 and R_1, $S = \mathrm{Spec}(R_0)$ and $X \xrightarrow{\xi} S$ an S-prescheme.

Let $\mathfrak{M}$ be the set of pairs (φ, τ), where $\varphi : X \to \mathrm{Proj}(R_\bullet)$ is a morphism of S-preschemes and $\tau : \varphi^* \mathcal{O}(1)_{\mathrm{Proj}(R_\bullet)} \cong \mathcal{O}_X$ a trivialization of $\varphi^*(\mathcal{O}(1))$. Assume $X = \mathrm{Spec}(A)$, so $\xi : \mathrm{Spec}(A) \to \mathrm{Spec}(R_0)$ corresponds to $R_0 \to A$.

Let $\mathfrak{N}$ be the set of all surjective morphisms

$$A \otimes_{R_0} R_\bullet \xrightarrow{\alpha} A[T]$$

of $\mathbb{N}$-graded ~~Al~~ A-algebras where $\deg(T) = 1$.

We have a bijection $\mathfrak{M} \cong \mathfrak{N}$ which is compatible with restrictions to affine open subsets of X, defined as follows:

To $\alpha \in \mathfrak{N}$, we associate the morphism

$$\varphi : X \cong \mathbb{P}^0_A \cong \mathrm{Proj}(A[T]) \longrightarrow \mathrm{Proj}(A \otimes_{R_0} R_\bullet) \to \mathrm{Proj}(R_\bullet)$$

and the isomorphism

$$\tau : \varphi^* \mathcal{O}(1)_{\mathrm{Proj}(R_\bullet)} \cong \mathcal{O}(1)_{\mathbb{P}^0_X} \cong \mathcal{O}_X$$

Let $(\varphi, \tau) \in \mathfrak{M}$. Every element $r \in R_k$ defines a global section

$$[r] \in (\mathcal{O}_{\mathrm{Proj}(R_\bullet)}(k))(\mathrm{Proj}(R_\bullet))$$

and we put

$$\alpha(a \otimes r) = a \cdot \tau^{\otimes k}([r]) \cdot T^k$$

to obtain $\alpha \in \mathfrak{N}$.

Then these maps are well-defined and two-sided inverses.

Proof: Relatively easy to check.

Proposition 23.1: Let S be any prescheme, $X \xrightarrow{\xi} S$ any S-prescheme and $\mathcal{L}$ a line bundle on X and $\mathcal{R}$ a q.c. graded $\mathcal{O}_S$-module generated by $\mathcal{R}_0$ and $\mathcal{R}_1$.

Let $\mathfrak{M}(X)$ be the set of pairs (φ, τ), where $X \xrightarrow{\varphi} \mathrm{Proj}(\mathcal{R}_\bullet)$ is a morphism of S-preschemes and $\varphi^* \mathcal{O}(1)_{\mathrm{Proj}(\mathcal{R}_\bullet)} \xrightarrow[\tau]{\cong} \mathcal{L}$ an isomorphism of line bundles.

Let $\mathfrak{N}(X)$ be the set of all graded $\mathcal{O}_X$-algebra morphisms

$$\xi^* \mathcal{R}_\bullet \longrightarrow \bigoplus_{k=0}^{\infty} \mathcal{L}^{\otimes k} \cdot T^k$$ inducing surjections on stalks.

Then we obtain a bijection $m(X) \xrightarrow{\cong} n(X)$ by associating to (φ, λ) the morphism

$$\xi^* R_k \cong \varphi^* \pi^* R_k \xrightarrow{\varphi^*(\varepsilon)} \varphi^* \mathcal{O}(k) \cong \varphi^* \mathcal{O}(1)^{\otimes k} \xrightarrow{\lambda^{\otimes k}} \mathcal{L}^k.$$

where $\pi : \underline{\mathrm{Proj}}(R_\cdot) \longrightarrow X$ is the projection, $\varepsilon : \pi^* R_k \to \mathcal{O}(k)$ coming from $R_k \longrightarrow \pi_* \mathcal{O}(k)$, sending $r \in R_k(U)$ to ~~the~~ $(\text{image of } r \text{ in } (R_x)_p)_{(x,p) \in \pi^{-1} U}$

<u>Proof</u> As $U \mapsto m(U)$ and $U \mapsto n(U)$ are sheaves of sets, it is sufficient to check this on an appropriate topology base of X.
We use the affine open subsets U s.t. $\mathcal{L}|U$ is trivial and $\xi(U)$ is contained in some affine open subset of S, s.t. Lemma 2.3.1 may be applied.

<u>Corollary 2.3.1</u>: $\mathrm{Proj}(R_\cdot)(X) := \mathrm{Hom}_{S\text{-presch}}(X, \mathrm{Proj}(R_\cdot))$ is in canonical bijection with the set of pairs $(\mathcal{L}, \alpha)$ where $\mathcal{L}$ is a line bundle of X and α a morphism

$$(4) \qquad \alpha : \xi^* R_\cdot \longrightarrow \bigoplus_{k=0}^{\infty} \mathcal{L}^{\otimes k} \cdot T^k$$

of graded $\mathcal{O}_X$-algebras defining surjections on stalks.

<u>Corollary 2.3.2</u>: Under the assumptions of the proposition on $R_\cdot$, let $T \xrightarrow{\tau} S$ be an S-prescheme. Then there is a canonical isomorphism

$$\underline{\mathrm{Proj}}_T(\tau^* R_\cdot) \xrightarrow{\cong} (\underline{\mathrm{Proj}}_S(R_\cdot)) \times_S T$$

defined by the projection $\underline{\mathrm{Proj}}_T(\tau^* R_\cdot) \longrightarrow T$ and the morphism $\underline{\mathrm{Proj}}_T(\tau^* R_\cdot) \longrightarrow \underline{\mathrm{Proj}}_S(R_\cdot)$ (over S) defined by proposition 2.3.1 and the morphism

$$(\tau \pi_T)^* R_k \cong \pi_T^* \tau^* R_k \xrightarrow{\varepsilon} \mathcal{O}(k)_{\mathrm{Proj}_T(\tau^* R_\cdot)}$$

where ε is similar to the morphism ε occuring in the formulation of proposition 2.3.1.

<u>Proof</u>: One uses prop 2.3.1 to confirm the universal property of the fibre product for $\underline{\mathrm{Proj}}_T(\tau^* R_\cdot)$

<u>Remark</u> The iso actually also exists without the assumption that $R_\cdot$ is generated by R_1.

Let $R_\bullet$ be an $\mathbb{N}$-graded q.c. $\mathcal{O}_X$-algebra, $\mathcal{L}$ a line bundle on X and $R_\bullet^{(\mathcal{L})} := \bigoplus_{k=0}^{\infty} R_k \otimes \mathcal{L}^{\otimes k}$ with the algebra structure defined by

$$(r\otimes \ell)\cdot(\rho\otimes\lambda) = (r\cdot\rho)\otimes(\ell\otimes\lambda)$$

Corollary 23.3: There is a canonical isomorphism

$$\underline{\operatorname{Proj}}(R_\bullet^{(\mathcal{L})}) \xrightarrow{\ i\ } \underline{\operatorname{Proj}}(R_\bullet)$$

together with an isomorphism

$$i^*(\mathcal{O}(1))_R \otimes \pi^*_{\operatorname{Proj}(R^{(\mathcal{L})})\to S}\,\mathcal{L} \xrightarrow[\iota]{\cong} \mathcal{O}(1)_{R_\bullet^{(\mathcal{L})}}$$

s.t.

$$\pi^*_{\operatorname{Proj}(R^{(\mathcal{L})})}(R_1\otimes\mathcal{L}) \cong \pi^*_{\operatorname{Proj}(R^{(\mathcal{L})})}R_1 \otimes \pi^*_{\operatorname{Proj}(R^{(\mathcal{L})})}\mathcal{L}$$

$$\xrightarrow{i^*(\varepsilon_{\operatorname{Proj}(R_\bullet)})\otimes id} i^*\mathcal{O}(1)_{R_\bullet}\otimes \pi^*_{\operatorname{Proj}(R^{(\mathcal{L})})}\mathcal{L} \xrightarrow{\ \iota\ } \mathcal{O}(1)_{R_\bullet^{(\mathcal{L})}}$$

coincides with

$$\pi^*(R_1\otimes\mathcal{L}) \xrightarrow{\varepsilon_{R^{(\mathcal{L})}}} \mathcal{O}(1)_{R^{(\mathcal{L})}}$$

Proposition 23.2: Let $R_\bullet$ and $S_\bullet$ be $\mathbb{N}$-graded $\mathcal{O}_S$-algebras satisfying the assumptions of proposition 23.1 and let $T = \bigoplus_{k=0}^{\infty} T_k$, where $T_k = R_k \otimes S_k$ equiped with the product $(r\otimes s)(\rho\otimes\sigma) = (r\rho)\otimes(s\sigma)$.

Let $X = \operatorname{Proj}(R_\bullet)$, $Y = \operatorname{Proj}(S_\bullet)$, $Z = X\times_S Y$, with projections $X \xleftarrow{\pi_1} Z \xrightarrow{\pi_2} Y$ and $\zeta = \pi_X\pi_1 = \pi_Y\pi_2 : Z \to S$

$$\begin{tikzcd} Z \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, "\zeta"] & X \arrow[d, "\pi_X"] \\ Y \arrow[r, "\pi_Y"'] & S \end{tikzcd}$$

Let $\mathcal{L} = \pi_1^*\mathcal{O}(1)_X \otimes \pi_2^*\mathcal{O}(1)_Y$, a line bundle on Z. We have epimorphisms of $\mathcal{O}_Z$-modules from $\zeta^*T_k \cong \zeta^*R_k\otimes\zeta^*S_k$ to $\mathcal{L}^{\otimes k} \xrightarrow{\cong} \pi_1^*\mathcal{O}(k)_X \otimes \pi_2^*\mathcal{O}(k)_Y$ defined by the tensor product of $\pi_1^*(\pi_X^*R_k \xrightarrow{\varepsilon_X} \mathcal{O}(1)_X)$ and $\pi_2^*(\pi_Y^*S_k \xrightarrow{\varepsilon_Y} \mathcal{O}(k)_Y)$

These epimorphisms intertwine the product $\zeta^*T_j\times\zeta^*T_k \to \zeta^*T_{k+j}$ on T with the tensor multiplication $\mathcal{L}^{\otimes j}\otimes\mathcal{L}^{\otimes k} \to \mathcal{L}^{\otimes j+k}$

Application of proposition 24.1 gives a morphism $Z \to \operatorname{Proj}(T_\bullet)$ of S-preschemes.

The claim is that this is an isomorphism.

Example: We have a closed embedding $\mathbb{P}^m_R \otimes \mathbb{P}^n_R \to \mathbb{P}^{m+n+mn}_R$ defined by $R[X_0\ldots X_m]_j \otimes R[Y_0\ldots Y_n]_j \leftarrow R[Z_{ab}]_j$ sending Z_{ab} to $X_a\otimes Y_b$.

Proof: By confirming that for any $T \xrightarrow{I} S$ and any morphism $\pi^* \mathcal{T}_\circ \xrightarrow{\alpha} \bigoplus_{i=0}^{\infty} \mathcal{L}^k \cdot T^k$ of $\mathcal{O}_T$-algebras which is an epimorphism, one has a decomposition $\mathcal{L} \cong M \otimes N$ and epimorphisms $\pi^* R_\circ \xrightarrow{\beta} \bigoplus_{i=0}^{\infty} M^k \cdot T^k$ and $\pi^* S_\circ \xrightarrow{\gamma} \bigoplus_{i=0}^{\infty} N^k \cdot T^k$ of $\mathcal{O}_T$-modules which is an algebra homomorphism, whose tensor product is α.

Definition 2.3.1: a) A morphism $X \xrightarrow{\xi} S$ of preschemes is called projective if there is an $\mathbb{N}$-graded q.c. $\mathcal{O}_S$-algebra $R_\circ$ generated by R_1 which is a locally f.g. $\mathcal{O}_S$-module, and an isomorphism $X \cong \mathrm{Proj}(R_\circ)$ of S-preschemes.

b) A morphism $X \xrightarrow{\xi} S$ is called strongly projective if it has the form $X \xrightarrow{i} \mathbb{P}^n_S \longrightarrow S$ where i is a closed embedding.

Remark 2.3.1 a) is EGA-definition and b) is Hartshorne definition. It turns that b) implies a).

In between, there is a class of morphisms that factor as $X \longrightarrow \mathbb{P}(\mathcal{E}) \longrightarrow S$, where $\mathcal{E}$ is a vector bundle, $\mathbb{P}(\mathcal{E}) := \mathrm{Proj}(S_\circ(\mathcal{E}))$ where $\mathcal{E} \longrightarrow S_1(\mathcal{E}) = \mathcal{E}$ has the universal property for morphisms $\mathcal{E} \longrightarrow R_1$, where R is a q.c. and graded $\mathcal{O}_S$-module;

$$S_\circ(\mathcal{E}) = \bigoplus_{k=0}^{\infty} S^k \mathcal{E}$$

where $\mathcal{E}^{\otimes k} \longrightarrow S^k \mathcal{E}$ classifies symmetric k-linear forms on $\mathcal{E}$.

Fact 2.3.2 If $X \xrightarrow{\xi} Y$ is (strongly) projective, then for any Y-prescheme $\hat{Y} \longrightarrow Y$ also $\hat{X} := X \times_Y \hat{Y} \longrightarrow \hat{Y}$ is (strongly) proj. If $\hat{Y} \longrightarrow Y$ is also (strongly) projective, so is $\hat{X} \longrightarrow Y$.

Proposition 2.3.3: Let $i: X \longrightarrow Y$ be a closed immersion and $Y \xrightarrow{f} S$ a (strongly) projective morphism. Then $X \xrightarrow{fi} S$ is also (strongly) projective.

Proof: Clear for strong projectivity. Let $Y = \mathrm{Proj}(R_\circ)$, with $R_\circ$ as in definition 2.4.1 a), and let $I_n \subseteq R_n$ be the preimage of $f_* I_X(n) \subseteq f_* \mathcal{O}_Y(n)$, under the epimorphism $R_n \longrightarrow f_* \mathcal{O}_Y(n)$, where I_X is the sheaf of ideals defining the subprescheme $X \subseteq Y$.

Let $\hat{R}_\circ = R/I_\circ$ and $\hat{X} = \mathrm{Proj}(\hat{R})$. The morphism $R_\circ \longrightarrow \hat{R}$ defines a closed embedding $\hat{X} \longrightarrow Y$. We claim that

$\tilde X = X$ as subpreschemes of Y, proving the claim as then $\tilde X \longrightarrow S$ is projective.

Obviously $X \subseteq \tilde X$ as all sections of $\mathcal{I}_n$ define sections of $\mathcal{O}_Y(n)$ ~~not~~ vanishing on X, because of the definition.

That equality holds, is a local question, hence the following may be applied:

Lemma 2.3.3: Let $Y = \mathrm{Proj}(R_\cdot)$, $X \subseteq Y$ a closed subprescheme defined by the sheaf of ideals $\mathcal{I}_X \subseteq \mathcal{O}_Y$. Let $I_\cdot \subseteq R_\cdot$ be defined as in the previous proof. If $r \in R_k$ and $g \in (\mathcal{I}(k))(U)$, where $U := Y \setminus V(r)$, then there is $\ell \in \mathbb{N}$ and $f \in I_{k\ell}$ s.t. $g = \frac{[f]}{r^\ell}$

Proof Apply lemma 2.2.2 to the q.c. $\mathcal{O}_Y$-module $\mathcal{I}_X$ and the line bundle $\mathcal{O}$

Corollary 2.3.4: a) Every strongly projective morphism is projective.

b) If there is an ample line bundle on S, then a morphism $X \longrightarrow S$ ~~which~~ is projective iff it is strongly projective.

Proof: Clear by prop 2.3.3, since $\mathbb{P}^n_S = \mathrm{Proj}(\mathcal{O}_S[X_0 \dots X_n]) \longrightarrow S$ is projective.

b) Let $\mathcal{L}$ be an ample line bundle on S. Then there is $k \in \mathbb{N}$ s.t. $R_1 \otimes \mathcal{L}^k$ is generated by its global sections.

Let $X = \mathrm{Proj}(R_\cdot) \longrightarrow S$ be a projective morphism. By replacing $R_\cdot$ by $R_\cdot^{(\mathcal{L}^k)}$ (of corollary 2.3.3) we may assume R_1 to be generated by its global sections say $\xi_0, \dots, \xi_n \in R_1(S)$. Then

$$\mathcal{O}_S[X_0, \dots, X_n] \longrightarrow R_\cdot : X_i \longmapsto \xi_i$$

is a multiplicative epimorphism of $\mathbb{N}$-graded q.c. $\mathcal{O}_S$-algebras defining a closed embedding $X \longrightarrow \mathbb{P}^n_S$ of S-preschemes. Hence $X \longrightarrow S$ is strongly projective.

Proposition 2.3.4: Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be strongly projective. Then gf is strongly projective.

If f, g are projective and Z is q.c. and q.s., then gf is proj.

Proof of first assertion: We have closed embeddings $X \longrightarrow \mathbb{P}^m_Y$ and $Y \longrightarrow \mathbb{P}^n_Z$, defining $\mathbb{P}^m_Y \longrightarrow \mathbb{P}^m_{\mathbb{P}^n_Z} \triangleq \mathbb{P}^m_Z \times_Z \mathbb{P}^n_Z \xrightarrow{\sigma} \mathbb{P}^{n+m+(n\cdot m)}_Z$ where σ is a closed immersion. Thus we have closed immersion

$$X \longrightarrow \mathbb{P}^{n+m}_Z$$

The second assertion is harder and omitted.

Example: Let A be a q.c. $\mathcal{O}_S$-algebra and

$$R_k = \begin{cases} \mathcal{O}_S \cdot T^0 & k=0 \\ A \cdot T^k & k>0. \end{cases}$$

Then $\underline{\mathrm{Proj}}(R_\cdot) \cong \underline{\mathrm{Spec}}(A)$ as $V(T) = \emptyset$, and $\mathrm{Proj}(R) \setminus V(T) = \mathrm{Spec}((R_T)_0)$

Hence finite morphisms are projective.

Theorem 4: Let $X \xrightarrow{f} S$ be a projective morphism of locally Noetherian preschemes. Let M be a coherent $\mathcal{O}_X$-module. Then the $\mathcal{O}_S$-module $R^k f_* M$ is coherent for all $k \in \mathbb{N}$.

Proof: The assertion is local w.r.t. S so we may assume that $S = \mathrm{Spec}(A)$ is affine.

Then $X = \mathrm{Proj}(R_\cdot)$, where $R_0 = A$, $R_\cdot$ is gen. by R_0, R_1 and R_1 is a f.g. A-module. If $\xi_1, \dots, \xi_n \in R$ are generators over A, then we have a closed embedding

$$X \xrightarrow{i} \mathbb{P}^n_A$$

of A-preschemes as in the proof of corollary 2.4.4. We have $R^k f_* M = R^k p_* i_* M$ by formula (1.5.3), where $\mathbb{P}^n_A \xrightarrow{p} A$.

As $i_* M$ is coherent, $H^k(\mathbb{P}^n_A, i_* M)$ is a f.g. A-module, by Theorem 3, hence $R^k p_* i_* M$ is a coherent $\mathcal{O}_S$-module.

Proposition 2.3.5: Let $X \xrightarrow{f} Y \xrightarrow{g} S$ be morphisms s.t. g is separated and gf (strongly) projective. Then f is (strongly) projective.

Lemma 2.3.4 Let $\mathcal{C}$ be a class of morphism contained in the class of separated morphisms, stable under base change and s.t. $fi \in \mathcal{C}$ when $f \in \mathcal{C}$ and i a closed embedding. Then $gf \in \mathcal{C}$ and g separated implies $f \in \mathcal{C}$.

(Recall: If $X \xrightarrow{f} S$ separated, and $S \xrightarrow{\alpha} X$ s.t. $f\alpha = id_S$, then α is a closed embedding, as $\alpha = \ker(X \overset{id}{\underset{\alpha f}{\rightrightarrows}} X)$.

Proof: Consider

$$\begin{tikzcd} & X \times_Z Y \arrow[dr, "h"] & \\ X \arrow[ur, "i"] \arrow[dr, "gf"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

where $i = (id_X, f)$ and h the base-change of gf w.r.t. $Y \xrightarrow{g} Z$. Then i is a closed embedding, since it's a section of the separated (base change of g) morphism $X \times_Z Y \xrightarrow{\pi_1} X$, and $h \in C$ as C is closed under base-change. It follows that $f = hi \in C$.

2.~~5~~4 Proper morphisms

Definition 2.4.1: Let $f: X \to Y$ be a morphism of preschemes, where Y is locally Noetherian.
We say that f is proper if it is of globally finite type, separated and universally closed.
Recall that f is closed if $f(C) \subseteq Y$ is closed for all closed $C \subseteq X$.
It is universally closed if all its base changes $\hat{X} = X \times_Y \hat{Y} \to \hat{Y}$ with any morphism $\hat{Y} \to Y$ is closed.

Remark: Obviously the classes of (universally) closed morphisms are stable under composition and are base local.
Universally closed morphisms are stable under base change.
It follows that the class of proper morphisms is base local and stable under composition and base-change.
Also, closed immersions and finite morphisms are proper.

Possible topics of Franke's course next semester

1. Jacobian of curves
2. Classical field theory
3. Applications to covering theory

Lemma 2.3.4 now gives

Proposition 2.4.1 The class of proper morphisms is base-local and stable under composition and base change.
If gf is proper and g separated, then f is proper.

Proposition 2.4.2: Let $R_\cdot$ be an $\mathbb{N}$-graded q.c. $\mathcal{O}_X$-algebra which is locally f.g. as an $\mathcal{O}_X$-algebra. Then
$\mathrm{Proj}(R_\cdot) \longrightarrow X$ is proper. In particular, projective morphisms are proper.

Proof: Question is local on X, so wma $X \cong \mathrm{Spec}\, A$.
We have to show that $\mathrm{Proj}(R_\cdot) \longrightarrow A$ is proper when $R_\cdot$ is an $\mathbb{N}$-graded A algebra of f.t.

We already know that $\mathrm{Proj}(R.)$ is a scheme of f.t. over $\mathrm{Spec}(A)$, so only universal closedness needs to be shown.
It is enough to show it is closed, because the assumptions are stable under base change, and the reduction to the affine case still works with this formulation as closedness is also base-local.
To show $\mathrm{Proj}(R.) \longrightarrow A$ is closed, let $R.$ be generated by $R_0, \dots, R_d$ as an A-algebra. Consider

$$(+) \quad I = \bigcup_{i=0}^{\infty} \bigcap_{j=i}^{\infty} \mathrm{Ann}_A(R_j)$$
$$= \bigcup_{i=0}^{\infty} \bigcap_{j=i}^{i+d} \mathrm{Ann}_A(R_j)$$

We will show that the image of $\mathrm{Proj}(R.)$ in $\mathrm{Spec}(A)$ equals $V(I)$, proving our claim.

<u>Fact</u>: If M is a f.g. module over a ring A,

$$\mathrm{supp}(M) = \{ \mathfrak{p} \in \mathrm{Spec}\, A \mid M_{\mathfrak{p}} \neq 0 \}$$
$$= V(\mathrm{Ann}_A(M))$$
$$= \{ \mathfrak{p} \in \mathrm{Spec} \mid \underbrace{M \otimes_A k(\mathfrak{p})}_{\cong\, M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}} \neq 0 \}$$

We thus have

$\mathfrak{p} \in V(I) \iff$ for all i, there is $j \in \{i, i+1, \dots, i+d\}$ s.t. $R_j \otimes k(\mathfrak{p}) \neq 0$

$\iff$ there are infinitely many j s.t. $R_j \otimes_A k(\mathfrak{p}) \neq 0$.

Since the image of $\mathrm{Proj}(R. \otimes_A k(\mathfrak{p})) = \mathrm{Proj}(R.) \times_{\mathrm{Spec}(A)} k(\mathfrak{p})$ in $\mathrm{Spec}\, A$ is contained in $\{\mathfrak{p}\}$, it is sufficient to let A be a field and show:

If $R.$ is an $\mathbb{N}$-graded k-algebra of f.t., then $\mathrm{Proj}(R.) \neq \emptyset$ iff there are infinitely many R_j s.t. $R_j \neq 0$

If $R_j = 0$ for $j \gg 0$, then $R_+ \subseteq \sqrt{\{0\}}$ implying $\mathrm{Proj}(R.) = \emptyset$.

On the other hand if $R.$ is generated $/k$ by $(r_i)_{i=1}^N$ which are homogeneous of degree $< d_*$ and $\mathrm{proj}(R.) = \emptyset$, then all r_i are nilpotent (i.e. $r_i^{e_i} = 0$) as otherwise

$$\emptyset \neq \mathrm{Spec}((R_{r_i})_0) \subseteq \mathrm{Proj}(R.)$$

Thus $R_j = 0$ when $j > \sum_{i=1}^{N} d \cdot e_i$

<u>Theorem 5</u>: Let $X \xrightarrow{f} Y$ be a proper morphism of locally Noetherian preschemes.

If $\mathcal{M}$ is a coherent $\mathcal{O}_X$-module and $p \in \mathbb{N}$, then $R^p f_* \mathcal{M}$ is coherent.

<u>Corollary</u>: If A is a Noetherian ring and X a proper A-scheme and $\mathcal{M}$ a coherent $\mathcal{O}_X$-modules, then $H^p(X, \mathcal{M})$ is a f.g. A-module.

<u>Idea of proof</u>: Use Chow's lemma to reduce to the case of a projective morphism.

<u>Chow's lemma</u>: For proper $X \xrightarrow{f} Y$, there are $\pi : \hat{X} \longrightarrow X$ s.t. $\pi^{-1}(U) \xrightarrow{\pi} U$ for some open dense $U \subseteq X$, s.t. $g := f\pi$ projective.

$$\begin{tikzcd}
\hat{X} \arrow[dr, "\pi"] \arrow[d, "g"'] & \\
\mathrm{Spec}\, A & X \arrow[l, "f"]
\end{tikzcd}$$

One first shows that $H^p(X, \pi_* \mathcal{N})$ is f.g. when $\mathcal{N}$ is a coherent $\mathcal{O}_{\hat{X}}$-module (using Leray SS & coherence of $R^p \pi_* \mathcal{N}$ and $R^p \pi|_U = 0$ when $p > 0$)

Then one uses that $\mathcal{M} \longrightarrow \pi_* \pi^* \mathcal{M}$ has vanishing kernel and cokernel outside U.

<u>Theorem 6</u>: Let $X \xrightarrow{f} \mathrm{Spec}\, A$ be a proper morphism.

For a line bundle $\mathcal{L}$ on X, t.f.a.e.

a) $\mathcal{L}$ is ample

b) Some power $\mathcal{L}^{\otimes k}$ of $\mathcal{L}$ is very <u>ample</u>

[Here $\mathcal{L}$ is <u>very ample</u> if it is generated by global sections $\ell_0 \dots \ell_n$ and that the morphism $X \longrightarrow \mathbb{P}^n_A$ defined by

$$\mathcal{O}_X[T_0 \dots T_n] \longrightarrow \bigoplus_{\nu=0}^{\infty} \mathcal{L}^{\nu} \quad : T_i \mapsto \ell_i$$

is a closed embedding.]

c) There is $k \in \mathbb{N}$ s.t. $\mathcal{L}^{\otimes k}$ is generated by global sections $\ell_0 \dots \ell_n$ and the above morphism $X \to \mathbb{P}^n_A$ is finite (or equivalently: affine)

d) If M is a coherent $\mathcal{O}_X$-module and $p > 0$, then $H^p(X, M \otimes \mathcal{L}^k) = 0$ when $p > 0$, $k \gg 0$.

3 Cohomology of Curves

Let X be a one-dimensional regular integral Noetherian, q.c. scheme. Let X_1 be the set of codimension 1 (closed) points and $\mathrm{Div}(X)$ the free abelian group generated by X_1.

For $D \in \mathrm{Div}(X)$, we have $\mathcal{O}_X(D)$ defined by

$$\mathcal{O}_X(D)(U) = \{ f \in K \mid v_x(f) \geq D(x) \}$$

This is a line bundle on X.

Here $K = \mathcal{O}_{X,\eta}$ is the field of "rational functions" on X, where η is the generic point of X. Also $v_x : K \longrightarrow \mathbb{Z} \cup \{\infty\}$ the valuation of the DVR $\mathcal{O}_{X,x}$

Any line bundle is isomorphic to a line bundle of this form, and the divisor class of D is uniquely determined.

(The divisor class is its coset modulo

$$\{ \mathrm{div}(f) = \sum_{x \in X_1} v_x(f) \cdot x \mid f \in K^* \})$$

Let us assume that X is a k-scheme, where k is a field. We have seen the bijections

$$\{\text{line bundles on } X\}/\sim \iff \{\mathcal{O}_X^*\text{-torseurs}\}$$

$$\iff H^1(X, \mathcal{O}_X^*) = \varinjlim_{\mathcal{U}} \check{H}^1(\mathcal{U}, \mathcal{O}_X^*) \xrightarrow{\mathrm{dlog}} H^1(X, \Omega_{X/k})$$

Let c_1 : {line bundles over X}/$\cong$ $\longrightarrow H^1(X, \Omega_{X/k})$ be its composition, called the first Chern class.

3.1 Formulation of results

Let k be an algebraically closed field. All schemes are supposed to be k-schemes and $\Omega_X := \Omega_{X/k}$.

Theorem 2: Let $C \longrightarrow \operatorname{Spec} k$ be a proper morphism, where C is a regular curve.

a) There is a unique homomorphism (In particular: $\deg(D)$ independent of representative D)
$$\deg : \operatorname{Pic}(C) \longrightarrow \mathbb{Z}$$
(with $\operatorname{Pic}(C) = \{\text{line bundles on } C\}/\cong$ the Picard group) such that $\deg(\mathcal{O}_C(D)) = \deg(D) := \sum_{x \in C_1} D(x)$

b) $H^1(C, \Omega_C)$ is a one-dimensional k-vector space and there is a unique isomorphism
$$H^1(C, \Omega_C) \xrightarrow[\cong]{T} k$$
such that the diagram

$$(2) \qquad \begin{tikzcd} \operatorname{Pic}(C) \arrow[r, "c_1"] \arrow[d, "\deg"] & H^1(C, \Omega_C^1) \arrow[d, "T"] \\ \mathbb{Z} \arrow[r, "\cdot 1_k"] & k \end{tikzcd}$$

commutes

c) For any vector bundle $\mathcal{V}$ on C, the pairing
$$(3) \qquad H^0(C, \mathcal{V}) \times H^1(\mathcal{V}^* \otimes \Omega_C) \xrightarrow{m} H^1(\Omega_C) \xrightarrow[\cong]{T} k$$
is a non-degenerate pairing of fin. dim. k-vector spaces. Here m is defined by
$$H^0(C, \mathcal{V}) = \mathcal{V}(C) \longrightarrow \operatorname{Hom}(\mathcal{V}^* \otimes \Omega_C, \Omega_C)$$
$$v \longmapsto (\ell \otimes \omega \longmapsto \ell(v) \cdot \omega)$$

Remark b+c) are called Serre-Duality. When X is smooth, it holds over any field when Ω_C is replaced by
$$\omega_X = \det \Omega_X = \Lambda^{\dim(X)} \Omega_X$$
When $n = \dim(X) \geq 2$, a) will no longer deal with line bundles and c_1 is replaced by c_n.
When $k \neq \bar{k}$, theorem still holds when C/k is smooth ($\deg(D)$ redef.)

Theorem 8 (Riemann-Roch) Let $g = \dim(\Omega_C(C))$ denote the genus of C. Then

$$(4) \quad \chi(\mathcal{V}) := \dim(\mathcal{V}(C)) - \dim H^1(C, \mathcal{V})$$
$$= \deg(\det \mathcal{V}) + \dim(\mathcal{V}) \cdot (1-g)$$

where $\det \mathcal{V} = \Lambda^{\dim \mathcal{V}}(\mathcal{V})$ is a line bundle ~~of linear alternating f.~~ with a $(d = \dim(\mathcal{V}))$-alternating linear alternating form $\mathcal{V}^d \longrightarrow \det \mathcal{V}$ having the universal property for alternating d-linear forms $\mathcal{V}^d \xrightarrow{w} M$ with values in $\mathcal{O}_C$-modules.

By Serre duality we also have

$$(5) \quad \dim(\mathcal{V}(C)) - \dim(\mathrm{Hom}_{\mathcal{O}_C}(\mathcal{V}, \Omega_C))$$
$$= \deg(\det \mathcal{V}) + \dim(\mathcal{V})(1-g).$$

(Recall that in general $\chi(M) = \sum_{i=0}^{\dim X} (-1)^i \dim H^i(X, M)$).

Remark: The existence of $\det(\mathcal{V})$ is trivial when $\mathcal{V} = \mathcal{O}_C^n$, as $(\vec{f}_1, \ldots, \vec{f}_n) \longmapsto \det(\vec{f}_1, \ldots, \vec{f}_n)$ has the desired universal property:

$$w(f_1, \ldots, f_n) = \det(f_1, \ldots, f_n)\, w(e_1, \ldots, e_n)$$

where $e_j = (\delta_{ij})_{i=1}^n$.

In general the existence of a $\det(\mathcal{V})$ with the desired universal property is a local problem, hence the general case follows from this one.

The $\mathcal{O}^*$-torseur defined by $\det \mathcal{V}$ is the image of the $GL_n(\mathcal{O}_C)$-torseur defined by $\mathcal{V}$ under

$$GL_n(\mathcal{O}_C) \xrightarrow{\det} \mathcal{O}_C^*.$$

Remark 3.1.1: For a line bundle $\mathcal{V} = \mathcal{L}$, the formula becomes

$$\chi(\mathcal{L}) = \deg(\mathcal{L}) + 1 - g$$

Hence

$$(6) \quad \ell(D) - \ell(K-D) = \deg D + 1 - g$$

(putting $\mathcal{L} = \mathcal{O}_C(D)$), where $\ell(D) := \dim(\mathcal{O}_C(D))$ for any divisor D, and where K is any (canonical) divisor such that $\mathcal{O}_C(K) \cong \Omega_C$.

We have

$$k \xrightarrow{\cong} \mathcal{O}_C(C)$$

as $\mathcal{O}_C(C)$ is a finite (thm 5) field extension (C being integral) of k, but by algebraic closedness of k all finite field extensions of k are trivial.

So (6) is correct for $D=0$ (i.e. $\ell(0)=1$).

For $D=K$, we obtain $g-1 = \deg K + 1 - g$, or

<u>Fact 3.1.1</u>: $\deg K = \deg \Omega_C = 2g-2$.

(If $g=0$, $C \cong \mathbb{P}^1$, $g=1$ are elleptic curves)

<u>Corollary 3.1.1</u> $\ell(D) \geq \deg D + 1 - g$ (Riemann's inequality)

<u>Example</u> If $C \xrightarrow{i} \mathbb{P}^2$ is a regular curve of degree d in $\mathbb{P}^2$, then it is given as the vanishing scheme of an irreducible polynomial homogeneous $P \in k[X,Y,Z]$ of degree d. Let $\mathcal{J}$ be the sheaf of ideals in $\mathcal{O}_{\mathbb{P}^2}$ defining C. We have $\mathcal{O}_{\mathbb{P}^2}(-d) \underset{\cong}{\xrightarrow{P}} \mathcal{J}$.

We have a ses

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow i_* \mathcal{O}_C \longrightarrow 0.$$

As i is an affine morphism, we have

$$H^p(C, \mathcal{O}_C) \cong H^p(\mathbb{P}^2, i_* \mathcal{O}_C).$$

hence

$$\chi(C, \mathcal{O}_C) = \chi(\mathbb{P}^2, i_* \mathcal{O}_C)$$

and

$$1-g = \chi(\mathcal{O}_C) = \chi(\mathbb{P}^2, i_* \mathcal{O}_C) = \chi(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) - \chi(\mathcal{J})$$

$$= 1 - \chi(\mathbb{P}^2, \mathcal{O}(-d)) = 1 - \dim(\mathbb{P}^2, \mathcal{O}(-d))$$

$$\overset{(\text{thm } 2)}{=} 1 + \dim H^2(\mathbb{P}^2, \mathcal{O}(d-3))$$

$$= 1 + P_2(d) = \frac{(d-1)(d+2)}{2}$$

Note the assumption of C being ~~projector~~ regular!

<u>3.2 First proofs</u>. We start with the proof of

$$\boxed{(\tilde{6}) \quad \chi(\mathcal{O}_C(D)) = \deg(D) + 1 - g.}$$

(We will always consider 1-dimensional proper Spec k-schemes which are regular and integral.)

This already ~~applies~~ implies theorem 7(a), since $\chi(\mathcal{O}_C(D))$ is independent of representative D.

Also formula (6) will follow once Serre duality (theorem 7c)
has been established.

The general Riemann-Roch formula for vector bundles on a curve also follows, because every vector bundle on a regular curve has a filtration

$$0 = \mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \dots \subseteq \mathcal{V}_\ell = \mathcal{V}$$

whose quotients are line bundles and both sides of (4) are additive for s.e.s.

$$0 \to \mathcal{V} \to \mathcal{U} \to \mathcal{W} \to 0$$

Since $\chi(\mathcal{U}) = \chi(\mathcal{V}) + \chi(\mathcal{W})$, $\dim \mathcal{U} = \dim \mathcal{V} + \dim \mathcal{W}$ and $\det \mathcal{U} \cong (\det \mathcal{V}) \otimes \det(\mathcal{W})$, since universal forms $\omega_{\mathcal{V}}$ on $\mathcal{V}$ and $\omega_{\mathcal{U}}$ on $\mathcal{U}$ gives a form. a form on $\mathcal{W}$ locally given by

$$\omega_{\mathcal{W}}(w_1, \dots, w_k) = \frac{\omega_{\mathcal{U}}(v_1, \dots, v_\ell, \hat{w}_1, \dots, \hat{w}_k)}{\omega_{\mathcal{V}}(v_1, \dots, v_\ell)}$$

where $\hat{w}_i$ is a section of $\mathcal{U}$ lifting w_i and the $(v_i)_{i=1}^{\ell}$ form a base of $\mathcal{V}$ (which we can both always do locally.)

Thus the RR-formula (4) will follow for general vector bundles and (5) will follow by Serre duality.

For $(\hat{6})$, note that

<u>Fact 3.2.1</u>: Let P denote the divisor defined by the closed point P of C, and $\mathcal{L}(\pm P) := \mathcal{L} \otimes \mathcal{O}_C(\pm P)$. Then

$$\chi(\mathcal{L}(\pm P)) = \chi(\mathcal{L}) \pm 1$$

hence $(\hat{6})$ holds for $\mathcal{L}(\pm P)$ iff it holds for $\mathcal{L}$.

<u>Proof</u>: Let $\operatorname{Spec} k = \{P\} \xrightarrow{i} C$ be the closed embedding, $\lambda \in \mathcal{L}^*(\mathcal{U})$ where $\mathcal{U}$ is a neighborhood of P and π a free generator of the sheaf of ideals $\mathcal{J}$ defining P on $\mathcal{U}$. In other words, $v_P(\pi) = 1$, $v_Q(\pi) = 0$ when $Q \in \mathcal{U} \setminus \{P\}$. Then we have a s.e.s.

$$0 \to \mathcal{L}(-P) \to \mathcal{L} \xrightarrow{\rho} i_* \mathcal{O}_{\operatorname{Spec} k} \to 0$$

where ρ is given by

$$\rho(\ell) = \left(\frac{\ell|_{V \cap \mathcal{U}}}{\lambda|_{V \cap \mathcal{U}}}\right)(P) \quad \text{when } \ell \in \mathcal{L}(V) \text{ s.t. } P \in V.$$

Similarly, we get

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}(p) \xrightarrow{q} i_* \mathcal{O}_{\operatorname{Spec} k} \longrightarrow 0$$

where

$$q(\ell) = \left(\frac{\ell|_{V\cap U} \cdot \pi|_{U\cap V}}{h\cdot|_{V\cap U}} \right)(p) \quad \text{for } \ell \in \mathcal{L}(p)(U) \text{ s.t. } p\in V$$

Hence $\chi(\mathcal{L}(\pm p)) = \chi(\mathcal{L}) \pm \chi(i_* \mathcal{O}_{\operatorname{Spec} k})$, by additivity of χ. But ~~d~~ as i is a closed immersion, it is an affine morphism so the Leray-SS collapses and

$$H^p(i_* \mathcal{O}_{\operatorname{Spec} k}) \cong H^p(\operatorname{Spec} k, \mathcal{O}_{\operatorname{Spec} k})$$

hence $\chi(i_* \mathcal{O}_{\operatorname{Spec} k}) = 1$ proving the claim.

Fact 3.2.2 Formula (7) holds for all divisors D.

Proof: For $D=0$ this is trivial. In general do induction on

$$\sum_{p\in C} |D(p)|.$$

using fact 3.2.1.

Our next preliminary step is to prove theorem 7 with $C = \mathbb{P}^1$. For theorem 7 (b), it suffices to provide an identification of $\mathcal{O}(-2)$ with Ω_C and showing that $G(\mathcal{O}(1))$ has a non-zero image in the one-dimensional (thm 2) $H^1(C, \Omega_C)$.

We have the Euler differential equation

$$X \frac{\partial f}{\partial X} + Y \frac{\partial f}{\partial Y} = d f$$

when f is a rational function ~~wit~~ which is homogeneous of degree d on an open ~~cover~~ cone in $\mathbb{A}^2_k$.

When $d=0$ (i.e. f defines a section of $\mathcal{O}_{\mathbb{P}^1}$) this becomes

$$\frac{1}{Y} \frac{\partial f}{\partial X} = - \frac{1}{X} \frac{\partial f}{\partial Y}.$$

hence we get a derivation

$$\mathcal{O}_{\mathbb{P}^1} \xrightarrow{d} \mathcal{O}(-2)$$

by

$$d(f) = \begin{cases} \frac{1}{Y} \frac{\partial f}{\partial X} & \text{on } U\setminus V(Y) \\ -\frac{1}{X} \frac{\partial f}{\partial Y} & \text{on } U\setminus V(X) \end{cases}$$

for $f \in \mathcal{O}_{\mathbb{P}^1}(U)$.

On $\mathbb{P}^1 \setminus V(Y)$, when $f(X,Y) = \varphi\left(\frac{X}{Y}\right)$, the upper line

gives $\frac{1}{Y^2}(\varphi')(\frac{X}{Y})$, hence d is a universal derivative there and similar considerations apply to $\mathbb{P}^1 \setminus V(X)$. Hence $\Omega^1_{\mathbb{P}^1} \cong \mathcal{O}(-2)$
(Not canonical.)

It remains to determine the image of $c_1(\mathcal{O}(1))$ under this identification followed by the isomorphism $H^1(\mathbb{P}^1_k, \mathcal{O}(-2)) \cong k$ from theorem 2 (d). Let $U_0 = \mathbb{P}^1_k \setminus V(X)$, $U_1 = \mathbb{P}^1_k \setminus V(Y)$, then $\mathcal{O}(1)$ is trivialized by X on U_0 and by Y on U_1. Let $\mathcal{U} : \mathbb{P}^1_k = U_0 \cup U_1$ denote this open cover. Reading carefully through the construction before proposition 1.7.1 again, we find that the element of $\check{H}^1_{alt}(\mathcal{U}, \mathcal{O}_C^*)$ which gives the $\mathcal{O}_C^*$-tensor $\mathcal{O}(1)^*$ is given by $\frac{X}{Y} \in \mathcal{O}_C^*(U_{01})$. The image of $f = \frac{X}{Y}$ under $f \longmapsto \frac{d(f)}{f}$ equals $\frac{1}{XY} \neq 0$, hence $c_1(\mathcal{O}(1))$ is mapped to a non-zero element.

Note that $df = \left(\frac{\partial f}{\partial x}\right)/Y = -\left(\frac{\partial f}{\partial y}\right)/X$.

The isomorphism $H^1(\mathbb{P}^1_k, \mathcal{O}(-2)) \cong k$ from theorem 2(d) sends $\frac{1}{XY}$ to ± 1, hence $c_1(\mathcal{O}(1))$ is sent to ± 1 by

$$H^1(\mathbb{P}^1, \Omega_{\mathbb{P}^1}) \cong H^1(\mathbb{P}^1, \mathcal{O}(-2)) \cong k.$$

Thus theorem 7 b) follows for $\mathbb{P}^1_k$, as does theorem 7 c) when $\mathcal{V} = \mathcal{O}(k)$ by the duality part of theorem 2.

By a result of Grothendieck (going back to Dedekind/Weber and Plemely) any vector bundle $\mathcal{V}$ on $\mathbb{P}^1_k$ can be written as a direct sum of line bundles $\mathcal{O}(k_i)$, showing theorem 7 (and theorem 8) for $C = \mathbb{P}^1_k$.

Strategy: Use a finite separable (inducing a separable extension of fields of rational functions) morphism $C \xrightarrow{f} \mathbb{P}^1_k$.
Then $H^*(C, \mathcal{V}) \xrightarrow{\cong} H^*(\mathbb{P}^1, f_* \mathcal{V})$, where $f_* \mathcal{V}$ is a vector bundle by the classification of modules over PID.
Thus $H^1(C, \mathcal{V}) \cong H^1(\mathbb{P}^1, f_* \mathcal{V}) \cong \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(f_* \mathcal{V}, \Omega_{\mathbb{P}^1})^*$
$\cong \mathrm{Hom}(\mathcal{V}, f^! \Omega_{\mathbb{P}^1})$, where $f^!$ (hopefully) is a right adjoint functor to f_*. Then, hopefully, one has a canonical isomorphism $f^! \Omega_{\mathbb{P}^1} \cong \Omega_C$. Then the fact that the

resulting isomorphism $H^1(C, \Omega_C) \longrightarrow k$ sends $G(\mathcal{O}(D))$ to the image of $\deg(D)$ in k also has to be checked.

3.3 The functor $f^!$

Let $C \xrightarrow{f} D$ be a finite separable morphism of regular connected curves over an algebraically closed field k. We want to construct a right adjoint (!)

$$f^! : \mathcal{O}_D\text{-Mod} \longrightarrow \mathcal{O}_C\text{-Mod}$$

Let R be a Dedekind domain with the field of fractions K and L/K a finite separable extension. Let S be the integral closure of R in L, $Tr_{L/K} : L \longrightarrow K$ the trace and $S^* = \{\sigma \in L \mid Tr_{L/K}(\sigma s) \in R \text{ for all } s \in S\}$. $\mathfrak{D}_{S/R} := (S^*)^{-1}$, the Dedekind different.

This commutes with localization, hence

Fact 3.3.1: If f is as above, there is a unique sheaf of ideals $\mathfrak{D}_f \subseteq \mathcal{O}_C$ s.t.

$$\mathfrak{D}_f(f^{-1}U) = \mathfrak{D}_{\mathcal{O}_C(f^{-1}U)/\mathcal{O}_D(U)}$$

for all open $U \subseteq D$.

We view $\mathfrak{D}_f^{-1}$ (the inverse of the line bundle $\mathfrak{D}_f$) as a subsheaf of $\mathcal{K}_C$, the (constant) sheaf of rational functions on C. By the definition

$K_C \xrightarrow{Tr_{K_C/K_D}} K_D$ defines

$$f_* \mathcal{K}_C \xrightarrow{Tr} \mathcal{K}_D \quad \text{and} \quad f_* \mathfrak{D}_f^{-1} \xrightarrow{Tr_{C/D}} \mathcal{O}_D .$$

If M is a q.c. $\mathcal{O}_D$-module, we denote by $Tr_{C/D}$:

$$f_*(f^*M \otimes_{\mathcal{O}_C} \mathfrak{D}_f^{-1}) \cong M \otimes_{\mathcal{O}_D} f_* \mathfrak{D}_f^{-1} \xrightarrow{id_M \otimes Tr_{C/D}} M \otimes_{\mathcal{O}_D} \mathcal{O}_D \cong M.$$

Here we use

$$f_*(f^*A \otimes B) \cong A \otimes f_* B$$

$$[A(U) \otimes_{\mathcal{O}_D(U)} \mathcal{O}_C(f^{-1}(U))] \otimes_{\mathcal{O}_C(f^{-1}(U))} B(f^{-1}(U)) \xrightarrow{\cong} A(U) \otimes_{\mathcal{O}_D(U)} B(f^{-1}(U))$$

$$(a \otimes r) \otimes b \longmapsto a \otimes (rb).$$

The above composition is called $\mathrm{Tr}_{C/D}$ or Tr_f.
As $S \times S^* \xrightarrow{\mathrm{Tr}_{L/K}} R$ is a non-degenerate duality in the above situation of Dedekind domains, one obtains:

<u>Proposition 33.1</u>: For every q.c. $\mathcal{O}_C$-module N, the map

$$(2) \qquad \mathrm{Hom}_{\mathcal{O}_C}(N, f^*M \otimes_{\mathcal{O}_C} \mathcal{D}_f^{-1}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_D}(f_*N, M)$$

$$\varphi \longmapsto \mathrm{Tr}_f \circ f_*(\varphi)$$

is a bijection. Thus the functor

$$f^! : \underline{Qc}(D) \longrightarrow \underline{Qc}(C)$$

$$M \longmapsto f^*M \otimes_C \mathcal{D}_f^{-1}$$

is right adjoint to $f_* : \underline{Qc}(C) \longrightarrow \underline{Qc}(D)$.

<u>Lemma 33.1</u>: Let f be as above, $d \in D$ any closed point and $c_1, \dots, c_N$ the preimages in C.
a) There is an affine neighborhood $\mathcal{U}$ of d and $\xi \in \mathcal{O}_C(f^{-1}\mathcal{U})$ s.t. $\xi(c_i) \neq \xi(c_j)$ and such that $\xi - \xi(c_i) \notin m_{C,c_i}^2 \subseteq \mathcal{O}_{C,c_i}$ for $1 \le i \le N$
b) If ξ has this property, we can shrink $\mathcal{U}$ such that $\mathcal{O}_C(f^{-1}\mathcal{U})$ is generated by ξ as $\mathcal{O}_D(\mathcal{U})$-algebra.

<u>Proof</u>: Existence of ξ is easy:

there are $\varphi_i \in \mathcal{O}_C(f^{-1}\mathcal{U})$ s.t. $\varphi_i(c_j) = \delta_{ij}$. Also let $\pi_i \in \mathcal{O}_C(f^{-1}\mathcal{U})$ whose image in $\mathcal{O}_{C,c_i}$ is a uniformizer. Then

$$\psi_i = \sum \varphi_i^2 (c_i + d_i \pi_i)$$

for which at c_i all summands with $j \neq i$ are in m_{C,c_i}^2, while the i-th is $c_i + c_i e_i \pi_i + d_i \pi_i$ when $\varphi_i^2 = 1 + e_i \pi_i \mod m_{C,c_i}^2$.
Choosing the c_i to be pairwise different and appropriate shows a).

b) To show that $(f_* \mathcal{O}_C)_d$ as an $\mathcal{O}_{C,c}$-algebra is generators by the powers of ξ (which implies the stated result but is also sufficient for our purpose) let $\mathfrak{m}_{C,c_i}^{e_i} = \mathfrak{m}_{D,d} \cdot \mathcal{O}_{C,c}$ (the e_i being the ramification indices), then the degree of field extensions is $d := \sum_{i=1}^{N} e_i$, and $(f_* \mathcal{O}_C)_d$ is a free $\mathcal{O}_{D,d}$-module of that rank.

In that module the elements $[\prod_{j=1}^{i-1} (\xi - c_j)^{e_j}](\xi - c_i)^k$ for $i = 1, \ldots, N$, $k = 0, \ldots, e_i - 1$ are k-linearly independent modulo $\mathfrak{m}_{D,d}(f_* \mathcal{O}_C)_d$, hence by NAK and a comparison of dimensions they generate $(f_* \mathcal{O}_C)_d$ as an $\mathcal{O}_{D,d}$-module.

<u>Proposition D</u>: Let R be a Dedekind domain, S its integral closure in a finite separable extension L of its field of quotients K, and $\xi \in S$ s.t. S is generated by ξ as an R-algebra, with $p(T) = T^n + \sum_{i=0}^{n-1} p_i T^i \in R[T]$ its minimum polynomial, $\mathfrak{D}(\xi) = p'(\xi)$, then $\mathfrak{D}_{S/R} = \mathfrak{D}(\xi) \cdot S$.

Using this in the situation of lemma 3.3.1 with $R = \mathcal{O}_D(U)$, $S = \mathcal{O}_C(f^{-1}U)$, and ξ from the lemma we obtain:

<u>Proposition 3.3.2</u>: If $C \xrightarrow{f} D$ is a finite separable morphism between regular curves over the algebraically closed field k, then $f^*\Omega_D \longrightarrow \Omega_C$ can be extended to a unique isomorphism $f^!\Omega_D = f^*\Omega_D \otimes \mathfrak{D}_f^{-1} \xrightarrow{\cong} \Omega_C$.
(Here $\Omega_C = \Omega_{C/k}$).

<u>Proof</u>: It follows from the above proposition and the short exact sequence of Kähler differentials that

$$\Omega_{S/R} = S/p'(\alpha)S \cong S/\mathfrak{D}_f$$

The second ~~result~~ short exact sequence together with a consideration of $d\xi$ shows the result.

For the proof of proposition D we need

Lemma: In the situation of proposition D, let

$$\frac{p(T)}{T-\xi} = \sum_{i=0}^{n-1} b_i T^i.$$

Then $\left(\frac{b_i}{p'(\xi)}\right)_{i=0}^{n-1}$ is a base of L/K, which is dual with respect to $(x,y) \longmapsto Tr_{L/K}(x \cdot y)$ to the base $(\xi^i)_{i=0}^{n-1}$.

Proof of proposition D: It follows from the lemma that for $S^* = \{\lambda \in L \mid Tr_{L/K}(\lambda S) \subseteq R\}$ we have

$$S^* = \frac{1}{p'(\alpha)} \sum_{i=0}^{n-1} b_i R$$

We have the relations $b_{n-1} = 1$, $b_{k-1} - \alpha b_k = p_k$ for $1 \leq k < n$, from which it follows by induction that

$$\sum_{i=0}^{k} \alpha^i_\xi R = \sum_{j=n-k-1}^{n-1} b_j R$$

Therefore the second factor in (+) is S:

$$S^* = \frac{S}{p'(\xi)}$$

(We use $\alpha = \xi$.)

Proof of the lemma: Let $\xi = \xi_1, \ldots, \xi_n$ be the $\neq$ images of ξ under the K-linear embeddings $L \xrightarrow{\sigma_i} \Sigma$, $(\sigma_i)_{i=1}^{n}$. Then for $\lambda \in L$, we have

$$Tr_{L/K}(\lambda) = \sum_{i=0}^{n-1} \sigma_i(\lambda)$$

We have

$$\sum_{i=1}^{n} \frac{p(T)}{T-\xi_i} \frac{\xi_i^r}{p'(\xi_i)} = T^r \qquad \text{for } r<n,$$

as both sides are polynomials of degree $<n$ in T and the equation holds when T is one of the ξ_i.

Thus $Tr_{L/K}\left(\frac{p(T)}{T-\xi} \xi^r\right) = T^r$ or

$$Tr_{L/K}(b_i \cdot \xi^r) = \delta_{ir}$$

as claimed.

Proposition 3.3.3: For the isomorphism $f^!\Omega_D \cong \Omega_C$ defined in the previous proposition, the following diagram commutes:

$$(4) \qquad \begin{tikzcd} f_* \mathcal{O}_C^* \arrow[r, "d\log"] \arrow[d, "N_{L/K}"] & f_* \Omega_C \cong f_* f^! \Omega_D \arrow[d, "Tr_f"] \\ \mathcal{O}_D^* \arrow[r, "d\log"] & \Omega_D \end{tikzcd}$$

(where we put $K := \mathcal{O}_{D,\eta_D}$, $L := \mathcal{O}_{C,\eta_C}$.)

Proof: Because all involved sheaves are vector bundles on C, sections on an open subset $U \neq \emptyset$ injectively map to the stalks of the generic point η_C. Therefore it is sufficient to show the following:

Let L/K be a finite separable extension, $k \subseteq K$ a subfield. As $\Omega_{L/K} = 0$ we have $\Omega_{L/k} \cong \Omega_{K/k} \otimes_K L$. Consider

$$\begin{tikzcd} \Omega_{L/k} \cong \Omega_{K/k} \otimes_K L \arrow[d, "Tr_{L/K}"] \\ \Omega_{K/k} \end{tikzcd}$$

Then for $\ell \in L^*$, $Tr_{L/K}(d\log(\ell)) = d\log(N_{L/K}(\ell))$.

If M is a normal hull of L/K, then it is still separable, and denoting $(\sigma_i)_{i=1}^n$ the K-linear embeddings $L \to M$, $n = [L:K]$, we have in $\Omega_{M/k} \cong \Omega_{K/k} \otimes_K M$ that

$$d_{K/k}\log(N_{L/K}(\lambda)) := \frac{d_{K/k}(N_{L/K}(\lambda))}{N_{L/K}(\lambda)}$$

$$= \frac{d_{K/k}\left(\prod_{i=1}^n \sigma_i(\lambda)\right)}{\prod_{i=1}^n \sigma_i(\lambda)}$$

$$= \sum_{i=1}^n \frac{d_{M/k}(\sigma_i(\lambda)) \cdot \prod_{j\neq i} \sigma_j(\lambda)}{\prod_{j=1}^n \sigma_j(\lambda)}$$

$$= \sum_{i=1}^n \frac{d_{M/k}(\sigma_i(\lambda))}{\sigma_i(\lambda)}$$

$$= \sum_{i=1}^{n} \sigma_i \left(\frac{d_{M/k}(\lambda)}{\lambda} \right) = \sum_{i=1}^{n} \sigma_i (d_{M/k} \log \lambda)$$

$$= \mathrm{Tr}_{L/K} \; d_{M/k} \log \lambda,$$

where the automorphism of L/K induced by σ_i was denoted by σ_i as well.

3.4 Proof of Serre Duality for Curves

Proposition 3.4.1: Let $f \in \mathcal{O}_{C,\eta_C}$ be a rational function on C s.t. $d_C f \neq 0$. Let $P = \{c \in C \mid f \text{ has a pole at } c\}$ be the set of poles of f. Then the morphism

$$C \setminus P \xrightarrow{f} \mathbb{A}^1 = \mathrm{Spec}\, k[T]$$

defined by

$$k[T] \longrightarrow \mathcal{O}_C(C \setminus P) : T \longmapsto f$$

extends to a unique finite separable morphism

$$C \xrightarrow{f} \mathbb{P}^1.$$

(In this subsection, we always let C be a proper regular curve over the algebraically closed field k.)

Proof: Let $C \setminus P \xrightarrow{f} \mathbb{A}^1$ be defined as above. We have

$$f^* d_{\mathbb{A}^1/k} T = df \neq 0,$$

from which one obtains $\Omega_{L/K} = 0$ where $L = \mathcal{O}_{C,\eta_C}$ and $K = k(T)$. Hence f is separable.

Let $N \subseteq C$ denote the set of zeroes of f. An extension of $f|_{C \setminus (N \cup P)} : C \setminus (N \cup P) \longrightarrow \mathbb{A}^1$ to a morphism $C \setminus N \longrightarrow \mathbb{P}^1$ is given by

$$C \setminus N \xrightarrow{1/f} \mathbb{A}^1_k \longrightarrow \mathbb{P}^1_k.$$

where the initial embedding $\mathbb{A}^1_k \longrightarrow \mathbb{P}^1_k$ (applied to f) was $\mathrm{Spec}\,(k[X,Y][\frac{1}{Y}])_0 \longrightarrow \mathbb{P}^1_k$ and the second one (applied to $1/f$) is $\mathrm{Spec}\, k[Y] = \mathrm{Spec}(k[X,Y][\frac{1}{X}])_0 \longrightarrow \mathbb{P}^1_k$. The existence of the extension of f is therefore shown and the uniqueness follows because C is separated.

The finiteness of the resulting morphism follows from Zariski's main theorem, assuming C to be projective s.t. any finite subset of C is contained in an affine open subset as follows:

Let $d \in \mathbb{P}^1$, and $a_1, \dots, a_n$ its preimages in C. Let $U \subseteq C$ be an affine neighborhood of $\{a_1, \dots, a_n\}$ whose image in $\mathbb{P}^1$ is not ~~$\mathbb{A}$~~ $\mathbb{P}^1$. W.l.o.g. let the image of U in $\mathbb{P}^1$ be contained in $\mathbb{A}^1$. Let $t_1, \dots, t_k$ be an enumeration of $f(C \setminus U)$ in $\mathbb{A}^1$. Then d is different from all t_k.

$V := \mathbb{A}^1 \setminus \{t_1, \dots, t_k\} = \mathbb{A}^1 \setminus V(\prod_{j=1}^{k} (T - t_k))$ is affine as is $f^{-1}(V) = U \setminus V(\prod_{j=1}^{k} (f - t_j))$

Hence the affine open subsets V of $\mathbb{P}^1$ s.t. $f^{-1}(V)$ is affine cover $\mathbb{P}^1$, showing that f is affine and hence by theorem 5 ($f_* \mathcal{O}_C$ is coherent) also finite.

We are now able to apply the results of 3.3 to the morphism $C \xrightarrow{f} \mathbb{P}^1 =: D$, constructed in proposition 3.4.1

As Serre duality holds for $\mathbb{P}^1$, we have an isomorphism (depending on f but otherwise canonical)

$$\begin{aligned} H^1(C, \mathcal{V})^* &\cong H^1(D, f_* \mathcal{V})^* \\ &\cong \mathrm{Hom}_{\mathcal{O}_D}(f_* \mathcal{V}, \Omega_D) \\ &\cong \mathrm{Hom}_{\mathcal{O}_C}(\mathcal{V}, f^! \Omega_D) \\ &\overset{\text{Prop 3.3.2}}{\cong} \mathrm{Hom}_{\mathcal{O}_C}(\mathcal{V}, \Omega_C), \end{aligned}$$

proving a duality like Theorem 7c).

In particular for $\mathcal{V} = \Omega_C$, $H^1(C, \Omega_C) \cong k$, and it is sufficient to show that the resulting isomorphism $H^1(C, \Omega_C) \cong k$ satisfies Theorem 7a). But the morphism $H^1(C, \Omega_D) \cong k \cong H^1(D, \Omega_D)$ obtained by these ~~ob~~ identifications is the one given by $f_* \Omega_C \xrightarrow{\mathrm{Trf}} \Omega_D$, and the result may be obtained by an application of Proposition 3.3.3.

If Δ is any divisor on C, then the image under $N_{L/K}: f_* \mathcal{O}^* \to \mathcal{O}$

of the tensor $\mathcal{O}_C(\Delta)^*$ is the torseur $\mathcal{O}_D(f_*\Delta)^*$, and

$$\deg(f_*\Delta) = \deg(\Delta).$$

<u>Remark</u>: Let $C \xrightarrow{f} D$ be a finite morphism between regular connected curves over $\bar{k} = k$. For a proper closed subprescheme $Z \subseteq D$ and $d \in D$, let $\mathcal{J}_Z$ be the sheaf of ideals defining Z and let $\mathrm{Div}(Z)$ be $\sum_{d \in D} c_d \cdot d$ where $c_d = v_{\mathcal{O}_{D,d}}(f)$ where f is a generator of $\mathcal{J}$ in some neighborhood of d. Then $\mathcal{O}_D(-\mathrm{Div}(Z)) \cong \mathcal{J}_Z$ by the definitions.

If the pull-back of divisors is defined by

$$f^*\delta = \sum_{c \in C} e_f(c) \cdot \delta_{f(c)},$$

then

$$f^*\mathcal{O}_D(\delta) \cong \mathcal{O}_C(f^*\delta)$$

Here $e_c(f) = v_{\mathcal{O}_{C,c}}(f^*\pi)$, where π is a uniformizer of D at $f(c)$. (i.e. of $\mathcal{O}_{D,f(c)}$). From proposition 3.3.2 (i.e. $f^!\Omega_D \cong \Omega_C$) we obtain that

$$\begin{aligned}
2g_C - 2 &= \deg(\Omega_C) = \deg(f^!\Omega_D) \\
&= \deg(\mathcal{D}_f^{-1} \otimes f^*\Omega_D) \\
&= \deg(\mathcal{D}_f^{-1}) + \deg(f^*\Omega_D) \\
(H) \qquad &= \deg(\mathcal{D}_f^{-1}) + \deg(f)\deg(\Omega_C) \\
&= \deg(\mathcal{D}_f^{-1}) + \deg(f)(2g_D - 2) \\
&= \Big(\sum_{c \in C} v_c(\mathcal{D}_f)\Big) + \deg(f)(2g_D - 2)
\end{aligned}$$

~~He~~ Which is the Hurewitz formula for the genus.

Here $\deg(f) := [\mathcal{O}_{C,\eta_C} : \mathcal{O}_{D,\eta_D}]$, i.e. the number n s.t. $f_*\mathcal{O}_C$ is an n-dimensional vector bundle on D.

The equation $\deg(f^*\delta) = \deg(f)\deg(\delta)$ follows from

$$\sum_{c \in f^{-1}(d)} e_c(f) = \deg(f).$$

(In fact, in general

$$\deg(f) = \sum_{c \in f^{-1}(d)} e_c(f)\,[k(c) : k(f(c))] \quad)$$

which is well-known and follows from the fact that $f_*\mathcal{O}_C$ is a vector bundle.

In the case that the ramification is tame (no $e_c(f)$ is divisible by the characteristic [and $k(c)/k(f(c))$ separable].) we have

$$v_c(\mathfrak{D}_f) = e_c(f) - 1$$

by a theorem of Dedekind (Dedekindscher Differentensatz) and formula (H) becomes

$$(H_t) \quad 2g_C - 2 = \deg(f)\cdot(2g_D - 2) + \sum_{c\in C}(e_c(f) - 1)$$

4. Cohomology and base change

4.1 Base change by a flat morphism

An R-module M is <u>flat</u> if $M\otimes_R -$ is an exact functor on R-mod. An R-algebra S is flat if it is flat as an R-module.

<u>Fact</u>: Let A be any ring, B a flat A-algebra, $X\to \operatorname{Spec} A$ a q.s. A-scheme, M a q.c. $\mathcal{O}_X$-module and

$$Y = X\times_{\operatorname{Spec} A}\operatorname{Spec} B \xrightarrow{\pi} X$$

Then $H^*(Y, \pi^*M) \cong H^*(X, M)\otimes_A B$.

<u>Proof</u> (sketch) If $X = \operatorname{Spec} R$, $Y = \operatorname{Spec}(R\otimes_A B)$ is also affine, and when $M = \widetilde{M}$ then $\pi^*M = \widetilde{M\otimes_A B}$ as

$$M \longmapsto M\otimes_A B = M\otimes_R (R\otimes_A B)$$

is left adjoint to the forgetful functor from $(S = R\otimes_A B)$-modules to R-modules.

Then

$$(\pi^*M)(Y) \cong M(X)\otimes_A B.$$

Applying this to the $U_{i_0\ldots i_p}$ of a finite affine open covering $\mathfrak{U}$ of X we obtain

$$\check{C}^*(\pi^{-1}\mathfrak{U}, \pi^*M) \cong \check{C}^*(\mathfrak{U}, M)\otimes_A B,$$

and the result follows from exactness of $-\otimes_A B$ (implying that $-\otimes_A B$ is compatible with taking cohomology.)

<u>Remark</u>: Flatness was only needed in final step. Otherwise we get a s.s.

$$E_2^{pq} = \operatorname{Tor}^A_{-p}(B, H^q(X, M)) \Longrightarrow H^{p+q}(Y, \pi^*M).$$

Remark: More generally, when $X \xrightarrow{f} Y$ is q.c. and separated and $\hat{Y} \xrightarrow{\pi} Y$ is flat and

$$\begin{array}{ccc} \hat{X} = \hat{Y} \times_Y X & \xrightarrow{\pi_X} & X \\ \downarrow \hat{f} & & \\ \hat{Y} & & \end{array}$$

we have

$$\pi^* R^p f_* M \cong R^p \hat{f}_* (\pi_X^* M),$$

by applying the previous result to the affine open subsets of Y and $\hat{Y}$.

For non-flat π, we get again a spectral sequence.

4.2 The theorem about formal functions

Let R be a local Noetherian ring and let $\wedge$ denote the completion w.r.t. its maximal ideal $\mathfrak{m}$. This is an exact functor on f.g. R-modules, which is a consequence of the Artin-Rees theorem.

Let $X \xrightarrow{\xi} \operatorname{Spec} R$ be a proper morphism and M a coherent $\mathcal{O}_X$-module.

Let $X_0 = X \times_{\operatorname{Spec}(R)} \operatorname{Spec}(k(x))$ and

$$X_n = X \times_{\operatorname{Spec}(R)} \operatorname{Spec}(R/\mathfrak{m}^{n+1})$$

its n-th infinitesimal thickening.

Then $X_0 \longrightarrow X_n$ and $X_n \longrightarrow X_{n+1}$ are closed embeddings defined by a nilpotent sheaf of ideals, hence are homeomorphisms.

Theorem (Grothendieck, about formal functions)

For M a coherent $\mathcal{O}_X$-module in above situation and $i_n : X_n \longrightarrow X$ the closed embedding

$$\widehat{H^p(X, M)} \cong \varprojlim_n H^p(X_n, i_n^* M)$$

Remark: More generally, when $X \xrightarrow{\xi} Y$ is proper and Y is locally Noetherian and M coherent, then $\widehat{(R^p \xi_* M)_y}$ is an inverse limit of the cohomology of infinitesimal thickenings of the fiber of ξ over y.

Corollary 4.2.1: When p is larger than the dimension of the fibers of ξ, $R^p f_* M$ vanishes (for coherent M.)

Corollary 4.2.2: (A very simple version of Zariski's main theorem) When $X \xrightarrow{f} Y$ is a proper morphism between locally Noetherian schemes whose fibers are zero dimensional (i.e. finite), then f is finite.

Remark: There are other versions where $X \xrightarrow{f} Y$ is quasi-finite and one obtains a factorization

$$X \xhookrightarrow{j} \overline{X} \xrightarrow{\pi} Y$$

where π is finite and j is an open embedding.

Franke knows several proofs for the theorem and prefers the one given in EGA.

One uses the finiteness result for the cohomology of proper morphisms to show that

$$\bigoplus_{i=0}^{\infty} H^p(X, \operatorname{Ker}(M \longrightarrow (i_n)_*(M|_{X_n}))\cdot T^k$$

is a f.g. module over the Rees-algebra

$$\bigoplus_{n=0}^{\infty} m^n \cdot T^n \subseteq R[T]$$

(Recall $H^p(X_n, M|X_n) \cong H^p(X, (i_n)_*(M|X_n))$.

Another proof, which is for example given in Hartshorne would be to use our calculations from 2.2 when $X = \mathbb{P}^n_R$, $M = \mathcal{O}(k)$, and then using the exactness of $\wedge$ (and/or Multay-Leffer type results) to obtain the result for other $\mathcal{O}_{\mathbb{P}^n_R}$-modules, then (not done by Hartshorne) using Chow's lemma.

4.3 Base change for a flat proper morphism

If $X \xrightarrow{f} \underset{\operatorname{Spec}(A)}{A}$ is a flat proper morphism and A a Noetherian, one obtains a finite complex $\mathcal{G}^*(v)$ of locally free A-modules s.t. $H^p(X \times_{\operatorname{Spec}(A)} \operatorname{Spec}(B), \pi^* v) \cong H^p(\mathcal{G}^*(v) \otimes_A B)$, where v is a vector bundle on X.

For instance, if v is a vector bundle on X and the cohomology of the fibers vanishes in cohomological degrees $\neq p$, then $R^q f_* v = 0$ when $q \neq p$ and $R^p f_* v$ is a vector bundle compatible with base change to any A-algebra B. §